

FINITE GROUPS OF AUTOMORPHISMS OF ENRIQUES SURFACES AND THE MATHIEU GROUP M_{12}

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ABSTRACT. An action of a group G on an Enriques surface S is called Mathieu if it acts on $H^0(2K_S)$ trivially and every element of order 2, 4 has Lefschetz number 4. A finite group G has a Mathieu action on some Enriques surface if and only if it is isomorphic to a subgroup of the symmetric group \mathfrak{S}_6 of degree 6 and the order $|G|$ is not divisible by 2^4 . Explicit Mathieu actions of the three groups \mathfrak{S}_5 , N_{72} and \mathfrak{A}_6 , together with non-Mathieu one of H_{192} , on polarized Enriques surfaces of degree 30, 18, 10 and 6, respectively, are constructed without Torelli type theorem to prove the if part.

A (holomorphic) action of a group on a $K3$ surface X is *symplectic* if it acts on $H^0(K_X) \simeq \mathbb{C}$ trivially. The finite groups which can act symplectically on $K3$ surfaces are classified in [18], relating with the Mathieu group M_{23} . There are exactly eleven maximal groups

$$(*) \quad L_2(7), \mathfrak{A}_6, \mathfrak{S}_5, M_{20}, F_{384}, \mathfrak{A}_{4,4}, T_{192}, H_{192}, N_{72}, M_9, T_{48}$$

among them. In this article we give a similar classification for Enriques surfaces, relating with the symmetric group \mathfrak{S}_6 of degree 6 embedded in the Mathieu group M_{12} .

A (minimal) Enriques surface S is a smooth complete algebraic surface with $q = h^1(\mathcal{O}_S) = 0$, $p_g = h^0(K_S) = 0$ and $2K_S \sim 0$. Equivalently, S is the quotient of a $K3$ surface X by a (fixed point) free involution ε . An action of a group on an Enriques surface S is *semi-symplectic* if it acts on $H^0(2K_S) \simeq \mathbb{C}$ trivially. If a finite automorphism σ is semi-symplectic, then we have $\text{ord}(\sigma) \leq 6$ (Corollary 4.7) and the Lefschetz number $L(\sigma)$ takes the following values (Proposition 4.8).

$\text{ord}(\sigma)$	\parallel	1	\parallel	2	\parallel	3	\parallel	4	\parallel	5	\parallel	6
$L(\sigma)$	\parallel	12	\parallel	$-4, -2, \dots, 12$	\parallel	3	\parallel	$2, 4$	\parallel	2	\parallel	$1, 3$

Although the Lefschetz number of a finite symplectic automorphism depends only on its order for $K3$ surfaces, the above table disproves the similar statement for semi-symplectic automorphisms of Enriques surfaces (see also Example 6, Remark 1.6 and Remark 3.2). In this paper we study the most

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natural subcases from the viewpoint of the small Mathieu group M_{12} . A classification of general semi-symplectic actions will be discussed elsewhere.

The Mathieu group M_{12} is a finite simple group of sporadic type. It acts on $\Omega_+ = \{1, \dots, 12\}$ quintuply transitively and is of order $12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 = 2^6 \cdot 3^3 \cdot 5 \cdot 11$. The stabilizer subgroup of M_{12} at a point $\star \in \Omega_+$ is denoted by M_{11} . The number of fixed points of the permutation action $M_{11} \curvearrowright \Omega_+$ depends only on the order of an element and is given as follows.

(**)	order	1	2	3	4	5	6	8	11
	number of fixed points	12	4	3	4	2	1	2	1

After Table (**), we make the following

Definition 1. A semi-symplectic action of a group G on an Enriques surface is *Mathieu* if the Lefschetz number of $g \in G$ depends only on $\text{ord}(g)$ and coincides with the lower row in table (**).

For an action of G to be Mathieu, it suffices for elements $g \in G$ of order 2 or 4 to have Lefschetz number 4 by Lemma 4.10. Mathieu actions are automatically effective by definition. Our main result is as follows.

Theorem 2. *For a finite group G , the following two conditions are equivalent to each other.*

- (1) G has a Mathieu action on some Enriques surface.
- (2) G can be embedded into the symmetric group \mathfrak{S}_6 and the order $|G|$ is not divisible by 2^4 .

The following examples are the key of the proof of (2) \Rightarrow (1).

Example 3. *Let X be the minimal resolution of the complete intersection*

$$\sum_{i < j} x_i x_j = \sum_{i < j} \frac{1}{x_i x_j} = 0$$

in \mathbb{P}^4 and S its quotient by the involution induced by the Cremona transformation $\varepsilon : (x_i) \mapsto (1/x_i)$. Then the natural action of \mathfrak{S}_5 , the third group of (), on S is (semi-symplectic and) Mathieu (§§1.2). (In fact, this Enriques surface is isomorphic to the surface of type VII in [16]. See Remark 1.4.)*

Example 4. *Let $S = X/\varepsilon$ be the quotient of the complete intersection*

$$X : x_i^2 - (1 + \sqrt{3})x_{i-1}x_{i+1} = y_i^2 - (1 - \sqrt{3})y_{i-1}y_{i+1}, \quad i = 0, 1, 2 \in \mathbb{Z}/3$$

in \mathbb{P}^5 by the involution $\varepsilon : (x : y) \mapsto (x : -y)$. A subgroup $C_3^2 : C_4$ of the Hessian group G_{216} acts on X linearly and also on S . This action extends to that of N_{72} , the ninth of (). The extended action is Mathieu (§§2.1).*

The rational functions $F_{00} = (y_0 - y_1)/(x_0 - x_1)$ and its square define elliptic fibrations $X \rightarrow \mathbb{P}^1$ and $S \rightarrow \mathbb{P}^1$ of the K3 and Enriques surface in the above example, respectively. The elliptic fibration $S \rightarrow \mathbb{P}^1$ has four singular fibers of type I_3 . Hence its Jacobian fibration is induced by the Hesse pencil and S has a semi-symplectic action of C_3^2 , the Mordell-Weil group.

Example 5. *The action of $C_3^2:C_4$ in Example 4 and the above C_3^2 generate a Mathieu action of \mathfrak{A}_6 , the second of $(*)$, on the Enriques surface S (§§2.2).*

The eighth group H_{192} of $(*)$, not a subgroup of M_{11} , has the following action.

Example 6. *The surface*

$$X : v^2w^2 + u^2w^2 + u^2v^2 + 1 + \sqrt{-1}(u^2 + v^2 + w^2 + u^2v^2w^2) = 0$$

of tri-degree $(2, 2, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is a smooth K3 surface with a symplectic action of the group H_{192} , where u, v, w are the inhomogeneous coordinates of three projective lines. The involution $\varepsilon : (u, v, w) \mapsto (-u, -v, -w)$ is free, commutes with the action and hence the quotient Enriques surface $S = X/\varepsilon$ has a semi-symplectic action of H_{192} .

The involutions induced by

$$(u, v, w) \mapsto (u, -v, -w) \quad \text{and} \quad (u, v, w) \mapsto (-\sqrt{-1}/u, -\sqrt{-1}/v, -\sqrt{-1}/w)$$

are Mathieu, but $(u, v, w) \mapsto (u, w, v)$ is not. (In fact the Lefschetz number equals 2 on S .) Thus the action of Example 6 is not Mathieu. But we can find two Mathieu sub-actions by $C_2 \times \mathfrak{A}_4$ and $C_2 \times C_4$, see [19] and also Section 3. Though neither is a subgroup of M_{11} , both are subgroups of \mathfrak{S}_6 , which is why the group \mathfrak{S}_6 appears in Theorem 2.

Theorem 7. *The two conditions in Theorem 2 are equivalent to the following:*

- (3) *G is a subgroup of one of the five maximal groups $\mathfrak{A}_6, \mathfrak{S}_5, N_{72}, C_2 \times \mathfrak{A}_4$ and $C_2 \times C_4$.*
- (4) *G has a small Mathieu representation (Definition 4.9 (1)) with $\dim V^G \geq 3$, its 2-Sylow subgroup is embeddable into \mathfrak{S}_6 and $G \not\cong Q_{12}$, the generalized quaternion group of order 12.*

Finally using these results in characteristic zero, we extend our classification to tame Mathieu actions in positive characteristic.

Theorem 8. *Let k be an algebraically closed field of positive characteristic $p > 0$. For a finite group G with $(|G|, p) = 1$, the two conditions in Theorem 2 are equivalent, that is, G has a Mathieu action on some Enriques surface over k if and only if the condition (2) holds.*

The construction of the paper is as follows. We prove (3) \Rightarrow (1) of Theorems 2 and 7 for the three groups $\mathfrak{S}_5, N_{72}, \mathfrak{A}_6$ in a refined form (Theorem 1.2) in Sections 1 and 2, where we intensively study the surfaces in Examples 3, 4, 5. Mathieu actions for the other groups $C_2 \times \mathfrak{A}_4$ and $C_2 \times C_4$ are constructed in Section 3. They are nothing but the actions studied in detail in [19], but here we give a slightly different treatment.

We give a preliminary study of semi-symplectic and Mathieu automorphisms in Section 4. In Section 5 we study groups with small Mathieu representations. Finally in Section 6 we prove the other implications of

Theorem 2 and Theorem 7 and complete the proofs. Especially in Subsection 6.2 we classify all finite groups satisfying the equivalent conditions of main theorems. In Section 7, we prove Theorem 8.

In Appendix A, an Enriques analogue of [17, Appendix], from which this article stems, is presented to give an alternative proof of Theorem 1.2. In Appendix B, yet another lattice-theoretic construction of an \mathfrak{A}_6 action using the result of [14, 15] is presented.

Notation and conventions.

Algebraic varieties X are considered over the complex number field \mathbb{C} , except Theorem 8 and Section 7 where it is over an algebraically closed field k of positive characteristic.

For a smooth variety X , K_X denotes the canonical divisor class.

Notation of finite groups follows [18]. In particular, C_n , D_{2n} , Q_{4n} , \mathfrak{S}_n , \mathfrak{A}_n denotes the cyclic group of order n , the dihedral group of order $2n$, the generalized quaternion group of order $4n$, the symmetric group of degree n , the alternating group of degree n respectively. The definition of some other groups will be recalled when it is necessary. For groups A and B , $A : B$ denotes a split extension with normal subgroup A .

The Mathieu group M_{24} acts on the operator domain Ω consisting of 24 points and the small Mathieu group M_{12} is the stabilizer of a dodecad, denoted by Ω_+ .

The symbol U denotes the rank 2 lattice given by the symmetric matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Root lattices A_n , D_n and E_n are considered negative definite. The lattice obtained from a lattice L by replacing the bilinear form (\cdot) with $r(\cdot)$, r being a rational number, is denoted by $L(r)$.

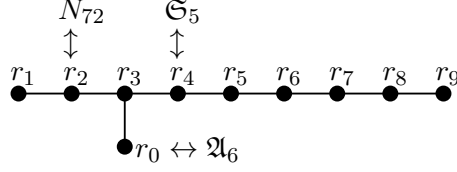
1. THE FIRST EXAMPLE OF MATHIEU ACTION

We recall that the symmetric group \mathfrak{S}_6 is a subgroup of M_{24} and decomposes the operator domain Ω into four orbits $\Omega_2, \Omega_{10}, \Omega_6, \Omega'_6$ of length 2, 10, 6 and 6 ([6]). Two permutation representations on the orbits Ω_6 and Ω'_6 of length 6 differ by the nontrivial outer automorphism of \mathfrak{S}_6 . The union $\Omega_6 \cup \Omega'_6$ is an umbral dodecad, and so is its complement $\Omega_2 \cup \Omega_{10}$. Thus \mathfrak{S}_6 is embedded into M_{12} in two different ways.

Among the eleven groups $(*)$, three groups $\mathfrak{S}_5, N_{72}, \mathfrak{A}_6$ are subgroups of \mathfrak{S}_6 . (The second one, $N_{72} \simeq C_3^2 : D_8$, is the normalizer of a 3-Sylow subgroup.) By the embedding $\mathfrak{S}_6 \hookrightarrow M_{24}$ above mentioned, they are also subgroups of M_{11} . More precisely we have the following:

Lemma 1.1. *Each of three groups $\mathfrak{S}_5, N_{72}, \mathfrak{A}_6$ is embedded into M_{11} so that it decomposes the operator domain $\Omega_+ \setminus \{\star\}$ into two orbits. The orbit length $\{a, b\}$ are $\{5, 6\}$, $\{2, 9\}$, $\{1, 10\}$, respectively.*

In fact, the three groups are isomorphic to the stabilizer of $\mathfrak{S}_6 \curvearrowright \Omega_i$ with $i = 6, 10, 2$, respectively.

FIGURE 1. The diagram $T_{2,3,7}$

As is well-known, the free part $H^2(S, \mathbb{Z})_f \simeq \mathbb{Z}^{10}$ of the second cohomology group of an Enriques surface S , equipped with the cup product, is isomorphic to the lattice associated with the diagram $T_{2,3,7}$ (Figure 1). The Weyl group of $T_{2,3,7}$ contains $\mathfrak{S}_5 \times \mathfrak{S}_6$, $\mathfrak{S}_2 \times \mathfrak{S}_9$ and \mathfrak{S}_{10} as Weyl subgroups. (The three subgroups become visible by removing the corresponding vertices shown in Figure 1.) Hence, via M_{11} and the Weyl groups, each of the three groups $G = \mathfrak{S}_5, N_{72}, \mathfrak{A}_6$ acts isometrically on $H^2(S, \mathbb{Z})_f$. The invariant part $H^2(S, \mathbb{Z})_f^G$ is generated by an element of square length ab and its orthogonal complement is isomorphic to the root lattice $A_{a-1} \oplus A_{b-1}$. In this and next sections, we construct a semi-symplectic action of G on an Enriques surface S which is not only Mathieu but also realizes the above G -action on $H^2(S, \mathbb{Z})_f$. Note that, since these groups are generated by involutions, the action is automatically semi-symplectic by Proposition 4.5.

Theorem 1.2. *The cohomological action $G \curvearrowright H^2(S, \mathbb{Z})_f$ of three groups in Examples 3, 4 and 5 are G -equivariantly isomorphic to the one described above. In particular, the actions $G \curvearrowright S$ in Examples 3, 4 and 5 are Mathieu, proving (3) \Rightarrow (1) of Theorems 2 and 7. Furthermore, the G -invariant (primitive) polarization is unique (up to numerical equivalence) and is of degree ab .*

We begin with some lattice-theoretic lemmas and then proceed to construct the group actions, proving the theorem for each group.

1.1. Some lattice theory. Let h be a nef and big divisor on an Enriques surface S . By [7, Lemma 2.9 and (2.11)], the quantity

$$\Phi(h) = \min\{(f, h) \mid f \in H^2(S, \mathbb{Z})_f \text{ is primitive with } (f^2) = 0 \text{ and } (f, h) > 0.\}$$

is attained by a nef isotropic element f . We call Φ the *gonality function* and a *gonality half-pencil* f for h is an isotropic element $f \in H^2(S, \mathbb{Z})_f$ satisfying $(f, h) = \Phi(h)$. We see that if h is a G -invariant polarization, G permutes gonality half-pencils for h .

Lemma 1.3. *We have the following.*

- (1) *Polarizations h with $(h^2) = 10$ and $\Phi(h) = 3$ are unique up to the orthogonal group $O(T_{2,3,7})$. Moreover, there are exactly ten gonality half-pencils for such h .*
- (2) *The same holds for $(h^2) = 18$ and $\Phi(h) = 4$. Moreover, there are exactly nine gonality half-pencils for such h .*

- (3) *The same holds for $(h^2) = 30$ and $\Phi(h) = 5$. Moreover, there are exactly six gonality half-pencils for such h .*

Proof. In each item, the former assertion is an easy consequence of the construction of the dual basis b_i ($0 \leq i \leq 9$) of r_i (in Figure 1) as in [7, (1.3)]. We have $h = b_0$, $h = b_2$ and $h = b_4$ respectively. For the latter assertions, we note the following decompositions of h into isotropic elements in terms of the isotropic sequence f_1, \dots, f_{10} :

$$b_0 = \frac{f_1 + \dots + f_{10}}{3}, \quad b_2 = \frac{(b_0 - f_1 - f_2) + f_3 + \dots + f_{10}}{2}, \quad b_4 = f_5 + \dots + f_{10}.$$

By Cauchy-Schwarz inequality, we see that the sets $\{f_i \mid 1 \leq i \leq 10\}$, $\{b_0 - f_1 - f_2, f_3, \dots, f_{10}\}$ and $\{f_i \mid 5 \leq i \leq 10\}$ give gonality half-pencils respectively. We note that the number of gonality half-pencils equals $\max\{a, b\}$ via Figure 1 where a, b are as in Lemma 1.1. \square

1.2. Mathieu action of \mathfrak{S}_5 . We consider the surface in \mathbb{P}^4 defined by

$$(1.1) \quad \overline{X}: \sum_{1 \leq i < j \leq 5} x_i x_j = \sum_{1 \leq i < j \leq 5} \frac{1}{x_i x_j} = 0,$$

which has five nodes at the coordinate points and whose minimal desingularization X is a K3 surface. The Cremona transformation $\varepsilon: (x_i) \mapsto (1/x_i)$ induces a free involution and we let S be the Enriques surface X/ε . The symmetric group \mathfrak{S}_5 acts on X and S by permutations of coordinates.

Proof of Theorem 1.2 for $G = \mathfrak{S}_5$. The key is to construct a good (rational) generator set of the second cohomology. The exceptional curves at nodes are interchanged with the rational curves $\overline{X} \cap \{x_i = 0\}$ by ε . Thus we get five smooth rational curves r_1, \dots, r_5 on S whose dual graph is the complete graph with doubled edges and five vertices, denoted $K_5^{[2]}$. Also, for each even involution $\sigma = (ij)(kl) \in \mathfrak{A}_5$, we can find lines

$$\begin{aligned} l'_\sigma: \quad x_i: x_j: x_k: x_l &= 1: -1: \sqrt{-1}: -\sqrt{-1}, \\ l''_\sigma: \quad x_i: x_j: x_k: x_l &= 1: -1: -\sqrt{-1}: \sqrt{-1}, \end{aligned}$$

lying in \overline{X} and interchanged by ε . Hence we have another 15 smooth rational curves l_σ on S . The incidence relation is given by

$$(l_\sigma, l_\tau) = \begin{cases} -2 & \text{if } \sigma = \tau, \\ 1 & \text{if } \sigma\tau \text{ has order 3,} \\ 0 & \text{otherwise,} \end{cases}$$

and their dual graph is isomorphic to the line graph¹ $L(P)$ of the famous Petersen graph P with 10 vertices and 15 edges. The connections between

¹The line graph, or edge graph, $L(\Gamma)$ of a graph Γ is the one whose vertices correspond to edges of Γ and two vertices are connected by an edge if they share a vertex in Γ .

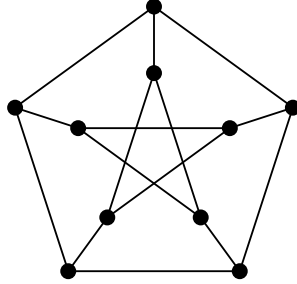


FIGURE 2. The Petersen graph

$K_5^{[2]}$ and $L(P)$ are given by

$$(r_i, l_\sigma) = \begin{cases} 2 & \text{if } \sigma(i) = i \\ 0 & \text{otherwise.} \end{cases}$$

The 20 curves in $K_5^{[2]} \cup L(P)$ generate $H^2(S, \mathbb{Z})_f$ up to index two. The overlattice structure is given, for example, by adding the half-pencil of the elliptic fibration defined by a pair of disjoint pentagons in $L(P)$. There are six such pairs in $L(P)$, hence we have six elliptic pencils on S with reducible fibers of type $I_5 + I_5$. We denote them by $|2f_j|$ ($1 \leq j \leq 6$). These classes satisfy $(f_i, f_j) = 1 - \delta_{ij}$.

Now, since the symplectic group action by \mathfrak{S}_5 is maximal (for $K3$ surfaces), we have the relation

$$\sum_{i=1}^5 r_i = \sum_{j=1}^6 f_j \in H^2(S, \mathbb{Z})_f$$

and this gives the \mathfrak{S}_5 -invariant polarization h of degree 30. Those f_j are exactly the gonality half-pencils for h as in the proof of Lemma 1.3. Moreover, the orthogonal complement of h is spanned by the nine elements

$$(r_i - r_{i+1})/2 \quad (1 \leq i \leq 4), \quad f_j - f_{j+1} \quad (1 \leq j \leq 5),$$

which is isomorphic to $A_4 + A_5$. The action of \mathfrak{S}_5 is isomorphic to that on the root lattice, therefore we obtain Theorem 1.2 for Example 3. (Every $r_i - r_{i+1}$ is divisible by 2 since $r_1 + r_2$ is equivalent to $2(l_{(12)(34)} + l_{(12)(35)} + l_{(12)(45)})$ and so on.)

Remark 1.4. We note that our Enriques surface S is isomorphic to the Enriques surface of type VII in [16]. In fact, the configuration of smooth rational curves we studied on S is the same as that of type VII. In particular, S has a finite automorphism group and by the main theorem of [16] we obtain the assertion. Our equation (1.1) will be valuable by its simplicity and ease to work with.

Remark 1.5. We can eliminate the variable x_5 from (1.1) and we get then the symmetric quartic surface $s_2^2 = s_1 s_3$, where s_i are the fundamental symmetric polynomials in x_1, \dots, x_4 . It has nodes at the coordinate points and has an obvious \mathfrak{S}_4 action, hence as in [20] we have a homomorphism

$$\varphi: \mathfrak{S}_4 \ltimes (C_2^{*4}) \rightarrow \text{Aut}(S),$$

where the generators of C_2 's are the covering involutions of the projection from coordinate points. Here we find that, opposed to the cases we treated in [20], φ is not an isomorphism. In fact, the four involutions corresponding to projections are nothing but the transpositions $(x_i x_5)$ and the image of φ is the finite group \mathfrak{S}_5 . In particular, it is not injective. By Remark 1.4, we see that $\text{Aut}(S) \simeq \mathfrak{S}_5$ and in fact φ is surjective.

Remark 1.6. Similar to (1.1), an Enriques surface with a (semi-symplectic) action of \mathfrak{S}_5 is obtained also from the quartic surface

$$(1.2) \quad \sum_{i=1}^5 x_i = \sum_{i=1}^5 \frac{1}{x_i} = 0.$$

The subaction of the alternating group \mathfrak{A}_5 is Mathieu but the full action is not Mathieu. (The quartic surface is the Hessian of the Clebsch diagonal cubic surface $\sum_{i=1}^5 x_i = \sum_{i=1}^5 x_i^3 = 0$, and this Enriques surface is Kondo's of type VI in [16]. See [9, Remark 2.4].)

2. ENRIQUES SURFACES OF HESSE-GODEAUX TYPE

In this section we prove Theorem 1.2 for N_{72} and \mathfrak{A}_6 . For that purpose we consider the $K3$ surface in \mathbb{P}^5

$$(2.1) \quad X = X_{\lambda, \mu}: \begin{cases} x_0^2 - \lambda x_1 x_2 = y_0^2 - \mu y_1 y_2 \\ x_1^2 - \lambda x_0 x_2 = y_1^2 - \mu y_0 y_2 \\ x_2^2 - \lambda x_0 x_1 = y_2^2 - \mu y_0 y_1 \end{cases} \quad (\lambda \neq \mu \text{ and } \lambda, \mu \neq 1, \omega, \omega^2),$$

where $\omega = (-1 + \sqrt{-3})/2$, and its Enriques quotient S by the free involution $\varepsilon: (x : y) \mapsto (x : -y)$. The abelian group C_3^2 acts on both X and S explicitly by

$$(2.2) \quad \begin{aligned} \alpha: (x_0 : x_1 : x_2 : y_0 : y_1 : y_2) &\mapsto (x_1 : x_2 : x_0 : y_1 : y_2 : y_0), \\ \beta: (x_0 : x_1 : x_2 : y_0 : y_1 : y_2) &\mapsto (x_0 : \omega x_1 : \omega^2 x_2 : y_0 : \omega y_1 : \omega^2 y_2). \end{aligned}$$

These surfaces have closer relation with the rational elliptic surface R given by the Hesse pencil of plane cubics

$$(2.3) \quad z_0^3 + z_1^3 + z_2^3 - 3\kappa z_0 z_1 z_2 = 0$$

and its two fibers at $\kappa = \lambda, \mu$. We use the following lemma.

Lemma 2.1. *Let $g: \mathbb{P}^2 \rightarrow \mathbb{P}^2$ be a collineation (namely a projective linear automorphism) preserving the Hesse pencil (2.3). It induces a Möbius transformation on the parameter κ of (2.3) and we denote it by the same*

letter g . Then, for any such g , we have an induced isomorphism between the Enriques surfaces of Hesse-Godeaux type

$$(2.4) \quad \begin{aligned} \tilde{g}: X_{\lambda, \mu} &\rightarrow X_{g(\lambda), g(\mu)} \\ (x : y) &\mapsto (c(\lambda, \mu)g(x) : g(y)) \end{aligned}$$

for a suitable constant $c(\lambda, \mu)$ depending on λ, μ and g .

Proof. We denote g by the matrix form $(g_{ij}) \in \mathrm{GL}(3, \mathbb{C})$ and use the notation $z' = g(z) \in \mathbb{P}^2$, $\kappa' = g(\kappa) \in \mathbb{P}^1$. We have the relation

$$z_0'^3 + z_1'^3 + z_2'^3 - 3\kappa' z_0' z_1' z_2' = c(\kappa)(z_0^3 + z_1^3 + z_2^3 - 3\kappa z_0 z_1 z_2)$$

where $c(\kappa)$ is a constant depending only on k and g . Differentiating both sides, we get

$$\sum_{i=0}^2 (z_i'^2 - \kappa' z_{i-1}' z_{i+1}') g_{ij} = c(k)(z_j^2 - \kappa z_{j-1} z_{j+1}) \quad (j = 0, 1, 2).$$

This equation applied to $\kappa = \lambda, \mu$ shows that $(x : y) \in X_{\lambda, \mu}$ if and only if $(g(x)/\sqrt{c(\lambda)} : g(y)/\sqrt{c(\mu)}) \in X_{g(\lambda), g(\mu)}$. \square

The group of collineations preserving the pencil (2.3) is the semi-direct product $C_3^2 : SL(2, \mathbb{F}_3)$ of the above C_3^2 by the binary octahedral group. It is of order 216 and called the *Hessian group*. We refer the readers to [5] for the explicit generators of this group.

In what follows we exhibit some elliptic pencils on X . First, let C_k ($k = 0, 1, 2$) be the conic on X defined by $x_i = y_i$ ($i \neq k$), $\lambda x_k = \mu y_k$. It is easy to see that the divisor $C := C_0 + C_1 + C_2$ constitutes a \mathbb{P}^1 -configuration of Kodaira type I_3 . For an element g in the Hessian group, let C_g be the pullback $\tilde{g}^*(C)$ of the corresponding configuration $C \subset X_{g(\lambda), g(\mu)}$ via Lemma 2.1, where g runs over the following

$$g = \begin{pmatrix} 1 & 1 & 1 \\ \omega & \omega^2 & 1 \\ \omega^2 & \omega & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & \omega \\ \omega & 1 & 1 \\ 1 & \omega & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & \omega^2 \\ \omega^2 & 1 & 1 \\ 1 & \omega^2 & 1 \end{pmatrix}, \text{ and } \mathrm{id}_R.$$

It turns out that these four I_3 configurations are disjoint each other and define an elliptic pencil F_∞ on X . It is invariant under ε , and we denote the induced pencil on S by $|2f_\infty|$. It has therefore 4 fibers of type I_3 and we call the 12 rational curves the *h-conics*. The pencil is called the *primary pencil*. Note that *h-conics* are stable under the action (2.2) of the group C_3^2 on X and S . These actions can be seen as induced from the Mordell-Weil group of R .

For the second pencils, we make use of the difference

$$(x_0 - x_1)(x_0 + x_1 + \lambda x_2) = (y_0 - y_1)(y_0 + y_1 + \mu y_2),$$

of the 1st and 2nd defining equations (2.1) of X . By this equality, the rational functions $F_{00} = (y_0 - y_1)/(x_0 - x_1)$ and $F'_{00} = (y_0 - y_1)/(x_0 + x_1 + \lambda x_2)$ are elliptic parameters on X and their squares $F_{00}^2, (F'_{00})^2$ define elliptic

pencils $|2f_{00}|, |2f'_{00}|$ on S . By applying the C_3^2 -action (2.2), we obtain 18 elliptic parameters on X as follows.

$$F_{kl} = \frac{y_k - \omega^l y_{k+1}}{x_k - \omega^l x_{k+1}}, \quad F'_{kl} = \frac{y_k - \omega^l y_{k+1}}{x_k + \omega^l x_{k+1} + \lambda \omega^{2l} x_{k+2}} (k, l \in \{0, 1, 2\} = \mathbb{Z}/3).$$

We denote by $|2f_{kl}|, |2f'_{kl}|$ the induced elliptic pencils on S .

Lemma 2.2. *We have the following relations between numerical classes.*
(1) $h = f_{kl} + f'_{kl}$, $(h, f_{kl}) = 2$, $(f_{kl}, f_{k'l'}) = 1 - \delta_{kk'}\delta_{ll'}$, where $\delta_{kk'}$ is the Kronecker delta and $h \in H^2(S, \mathbb{Z})_f$ is the natural polarization of degree 4. In particular, $\{h, f_{00}, \dots, f_{22}\}$ is a basis of the \mathbb{Q} -vector space $H^2(S, \mathbb{Q})$.
(2) $(h, f_\infty) = 3$ and $(f_\infty, f_{kl}) = 1$ ($\forall k, l$).

Proof. (1) and the first equality in (2) are deduced by definition. For the last equalities, it suffices to compute (f_∞, f_{00}) since f_∞ is invariant under the subgroup C_3^2 . We use the representative $F_\infty = C_0 + C_1 + C_2$. By definition of F_{00} , we see that $(F_{00}, C_2) = 0$ and $(F_{00}, C_i) = 1$ ($i = 0, 1$). Therefore, $(f_\infty, f_{00}) = (F_\infty, F_{00})/2 = \sum_{i=0}^2 (C_i, F_{00})/2 = 1$. \square

The 12 h -conics and 9 elliptic pencils $|2f_{00}|, \dots, |2f_{22}|$ have the following outstanding property.

Lemma 2.3. (1) *Every h -conic is contained as fiber in exactly three of the 9 elliptic fibrations defined by $|2f_{00}|, \dots, |2f_{22}|$.*
(2) *For every pair $|2f_{ij}|, |2f_{kl}|$ of elliptic pencils, there exists a unique h -conic which is contained in the fibers of elliptic pencils defined by them.*

The proof is a simple computation and we omit it. By this lemma, we obtain a Steiner system $\text{St}(2, 3, 9)$ on the set of 9 elliptic pencils, where the special triplet is the set of pencils containing a (fixed) h -conic.

An easy counting shows the following.

Corollary 2.4. *Every elliptic fibration defined by $|2f_{ij}|$ contains four h -conics in different fibers.*

For example, the elliptic parameter $(y_0 - y_1)^2/(x_0 - x_1)^2$ of $|2f_{00}|$ has four critical values

$$1, \frac{\lambda - 1}{\mu - 1}, \frac{\lambda - \omega}{\mu - \omega}, \frac{\lambda - \bar{\omega}}{\mu - \bar{\omega}},$$

at which the fiber contain an h -conic. Therefore, the elliptic pencils and the h -conics form the (Hesse) $(9_4, 12_3)$ -configuration.

2.1. Mathieu action of N_{72} . In this subsection we define a non-linear automorphism of Enriques surfaces of Hesse-Godeaux type to construct an N_{72} -action. For that purpose, we assume that

$$(2.5) \quad (\lambda + 1)(\mu + 1) = 1$$

holds and consider the matrix $A = (a_{kl})_{0 \leq k, l \leq 2}$ defined by

$$(2.6) \quad A = \begin{pmatrix} \mu x_0 & x_2 + cy_2 & x_1 - cy_1 \\ x_2 - cy_2 & \mu x_1 & x_0 + cy_0 \\ x_1 + cy_1 & x_0 - cy_0 & \mu x_2 \end{pmatrix},$$

where c is a constant satisfying $c^2 = 1 - \mu^2$. We regard (a_{kl}) as the homogeneous coordinates of \mathbb{P}^8 . Then the above expression (2.6) defines an embedding $\mathbb{P}^5 \subset \mathbb{P}^8$ whose linear equations are given by

$$2a_{kk} = \mu(a_{lm} + a_{ml}), \quad \{k, l, m\} = \{0, 1, 2\}.$$

Let δ be the correspondence $A \mapsto A^{\text{adj}} = (\tilde{a}_{kl})_{0 \leq k, l \leq 2}$, the adjoint matrix (namely, the transposed cofactor matrix) of A .² By $(A^{\text{adj}})^{\text{adj}} = (\det A)A$, δ defines a birational involution of \mathbb{P}^8 . Moreover, we can check that the defining equations (2.1) of the $K3$ surface X is nothing but the equations

$$2\tilde{a}_{kk} = \mu(\tilde{a}_{lm} + \tilde{a}_{ml}), \quad \{k, l, m\} = \{0, 1, 2\}$$

for the adjoint matrix A^{adj} . This shows that δ restricts to an involution on X . In terms of new coordinates, ε on \mathbb{P}^5 is identified with the involution $A \mapsto {}^t A$. Since the coordinate sets $\{x_i\}$ and $\{y_i\}$ form the basis of the positive and negative eigenspaces of this involution respectively, we see that δ commutes with ε and acts on S , too.

Now we consider the most special Enriques surface of Hesse-Godeaux type with parameter $\lambda, \mu = 1 \pm \sqrt{3}$, which satisfies (2.5). The fibers of (2.3) at $k = \lambda, \mu$ are preserved by the collineation

$$g = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}.$$

Via Lemma 2.1, it induces an automorphism of $X_{\lambda, \mu}$. Let γ be the one induced on S . The group of automorphisms on S generated by $\alpha, \beta, \gamma, \delta$ is denoted by G . It is easy to see that the linear automorphisms α, β, γ generate a subgroup $C_3^2 : C_4$ of order 36. To see the relations among α, β, γ and the non-linear δ , we note that the linear ones α, β and γ can be identified with the automorphisms $A \mapsto {}^t BAB$, where matrices B are given by

$$B_\alpha = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, B_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}, B_\gamma = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}.$$

From this observation we easily obtain the equalities

$$\delta\alpha = \alpha\delta, \quad \delta\beta = \beta^2\delta, \quad \delta\gamma = \gamma^{-1}\delta.$$

Thus, the group G is really the one $C_3^2 : D_8 \simeq N_{72}$.

² For example, the $(0, 1)$ -entry \tilde{a}_{01} is given by $-\mu x_2(x_2 + cy_2) + (x_0 - cy_0)(x_1 - cy_1)$.

Proof of Theorem 1.2 for $G = N_{72}$. Let us study the action of the Cremona involution δ on cohomology. Let I be the indeterminacy locus of δ as a transformation of \mathbb{P}^5 . It is obtained by cutting the Segre variety

$$\Sigma_{2,2} = \{A \in \mathbb{P}^8 \mid \text{rank } A \leq 1\}$$

three times by hyperplanes, hence is an elliptic curve of degree 6. (The smoothness of the linear section is easily seen by using two projections $I \rightarrow \mathbb{P}^2$.) In fact, since I and the curve $C_0 + C_1 + C_2$ are disjoint, I is linearly equivalent to the primary pencil F_∞ .

Since δ is defined by the linear system of quadrics containing I , we obtain $\delta^*(h) = 2h - f_\infty$. It follows that $\tilde{h} = h + \delta^*(h)$ is the polarization of degree 18 invariant under $G = N_{72}$. By looking at the intersection numbers with generators of $H^2(S, \mathbb{Q})$, we get $2f_\infty = -3h + \sum_{k,l} f_{kl}$ and we see

$$\tilde{h} = 3h - f_\infty = \frac{1}{2} \sum_{k,l} (h - f_{kl}) = \frac{1}{2} (f'_{00} + \cdots + f'_{22}).$$

Therefore, f'_{kl} ($0 \leq k, l \leq 2$) are the gonality half-pencils for \tilde{h} . The orthogonal complement of \tilde{h} is spanned by $h - f_\infty$ and the differences $f'_{kl} - f'_{k'l'}$ ($k, l, k', l' \in \{0, 1, 2\}$). They generate the lattice $A_1 + A_9$ and we obtain Theorem 1.2 for Example 4.

2.2. Mathieu action of \mathfrak{A}_6 . We continue to assume that $\lambda, \mu = 1 \pm \sqrt{3}$. As we saw the primary elliptic fibration $|2f_\infty|$ is of *Hesse type*, that is, has four singular fibers of type I_3 , and each of the nine elliptic fibrations $|2f_{00}|, \dots, |2f_{22}|$ has four reducible fibers (Corollary 2.4). On our special surface, each reducible fiber of $X \rightarrow \mathbb{P}^1$ becomes the union of a conic and two lines. An example of such line in X is given in parametrization by

$$(2.7) \quad (x_0 : x_1 : x_2 : y_0 : y_1 : y_2) \\ = ((1 - \sqrt{3})t : t + \sqrt{3} : t - \sqrt{3} : 1 + \sqrt{3} : \sqrt{3}t + 1 : -\sqrt{3}t + 1).$$

In fact, the elliptic parameters of $|2f_{01}|, |2f_{02}|, |2f_{21}|, |2f_{22}|$ takes constant values on this line. In total, 36 lines appear on X in such a way and we obtain 18 smooth rational curves on the Enriques surface S . We call them *h-lines*. These rational curves and 9 elliptic pencils $|2f_{00}|, \dots, |2f_{22}|$ satisfy the following:

Lemma 2.5. (1) *Every h-line is contained as fiber in exactly four of the 9 elliptic fibrations defined by $|2f_{00}|, \dots, |2f_{22}|$.*

(2) *For every triple $|2f_{ij}|, |2f_{kl}|, |2f_{mn}|$ of elliptic pencils, there exists a unique h-line which is contained in the fiber of elliptic pencils defined by them.*

Proposition 2.6. *Let T be the set of ten elliptic pencils $|2f_{ij}|$ ($0 \leq i, j \leq 2$) and the primary pencil $|2f_\infty|$. We call a 4-subset of T a *special quartet* if it is the set of pencils which contain a (fixed) h-conic or a (fixed) h-line. Then, the set of special quartets forms a Steiner system $\text{St}(3, 4, 10)$ on T .*

Proof. It follows from Lemmas 2.3 and 2.5. \square

Remark 2.7. The above 30 rational curves, say C_σ 's, are parametrized by the odd involutions σ in the symmetric group \mathfrak{S}_6 . There are two type of such involutions, 15 transpositions and 15 of permutation type $(2)^3$. The intersection number (C_σ, C_τ) is equal to 1 if σ and τ do not commute. When two distinct σ and τ commute, (C_σ, C_τ) is equal to 0 if their permutation types are the same and to 2 otherwise.

Let $f : S \rightarrow \mathbb{P}^1$ be an elliptic Enriques surface of Hesse type. The Jacobian fibration is a rational elliptic surface induced from the Hesse pencil (2.3). Its Mordell-Weil group is isomorphic to C_3^2 and acts on $f : S \rightarrow \mathbb{P}^1$ by translation. The following is the key for our construction of \mathfrak{A}_6 -action.

Lemma 2.8. *For an element $H \in H^2(S, \mathbb{Z})_f$, the following two conditions are equivalent:*

- (1) *H is invariant under the induced action of C_3^2 .*
- (2) *For every reducible fibre $C_1 + C_2 + C_3$ of f , we have*

$$(H.C_1) = (H.C_2) = (H.C_3).$$

Proof. Consider the two submodules A and B of $H^2(S, \mathbb{Z})_f$ consisting of H which satisfies (1) and (2), respectively, that is, the C_3^2 -invariant part A and the equal degree part B . Both $H^2(S, \mathbb{Z})_f/A$ and $H^2(S, \mathbb{Z})_f/B$ are free modules. Since the action of C_3^2 on $\{C_1, C_2, C_3\}$ is transitive on each of four reducible fibers, we have $A \subset B$. By the computation of character (see Proposition 4.8), A is of rank 2. Since the intersection $B \cap F^\perp$ is generated by F , where F is the fiber class, B is of rank at most 2. Hence we have $A = B$. \square

Now we focus on the new polarization $H = h + f_\infty$, which is of degree 10. All h -conics and h -lines are of degree 2 with respect to H . Hence, by the above lemma, the translations by Mordell-Weil groups of $|2f_{ij}|$ all preserve H . Since $f_\infty, f_{00}, \dots, f_{22}$ are the gonality half-pencils of H , the translation acts on the set $\{f_\infty, f_{00}, \dots, f_{22}\}$ by permutation (Lemma 1.3 and its preceding sentence). Let G be the automorphism group of (S, h) generated by all these translations. Then G contains 10 subgroups isomorphic to C_3^2 and preserves the Steiner system $\text{St}(3, 4, 10)$. Hence we have a surjective homomorphism $G \rightarrow \mathfrak{A}_6 \subset \text{Aut St}(3, 4, 10) \simeq \text{Aut}(\mathfrak{A}_6)$. Since \mathfrak{A}_6 is a maximal group which can act on a K3 surface symplectically, G is isomorphic to \mathfrak{A}_6 .

Proof of Theorem 1.2 for $G = \mathfrak{A}_6$.

The polarization $H = h + f_\infty$ on S is of degree 10 and the invariant lattice $H^2(S, \mathbb{Z})_f^{\mathfrak{A}_6}$ is spanned by it. The orthogonal complement of H in $H^2(S, \mathbb{Z})_f$ is generated by the differences of the gonality half-pencils f_{00}, \dots, f_{22} and f_∞ . Hence it is \mathfrak{A}_6 -equivariantly isomorphic to the A_9 -lattice and we obtain Theorem 1.2 for Example 5.

Remark 2.9. The intersection of two semi-symplectic actions of N_{72} and \mathfrak{A}_6 on S is $C_3^2 : C_4$. Hence we obtain a homomorphism $N_{72} * \mathfrak{A}_6 \rightarrow \text{Aut}^{ss} S$

from the amalgam over $C_3^2 : C_4$. In the forthcoming paper we will study the structure of $\text{Aut } S$ using this homomorphism and Remark 2.7.

3. MATHIEU ACTIONS OF $C_2 \times \mathfrak{A}_4$ AND $C_2 \times C_4$

We prove that (3) of Theorem 7 implies (1) of Theorem 2. By Theorem 1.2, it suffices to prove for two groups $C_2 \times \mathfrak{A}_4$ and $C_2 \times C_4$. We realize their Mathieu actions on the Enriques surface S in Example 6. The $K3$ -cover of S is a surface $X : F(u, v, w) = 0$ of tri-degree $(2, 2, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. A nonzero 2-form on X is obtained as residue of the rational 3-form $du \wedge dv \wedge dw / F(u, v, w)$. Hence X has the following symplectic actions.

- (a) The automorphism $(u, v, w) \mapsto (\pm u, \pm v, \pm w)$ with even number of ‘ -1 ’s is symplectic. Hence C_2^2 acts symplectically.
- (b) $(u, v, w) \mapsto (-\sqrt{-1}/u, -\sqrt{-1}/v, -\sqrt{-1}/w)$ is a symplectic involution.
- (c) For a permutation σ of $\{u, v, w\}$,

$$(u, v, w) \mapsto (\sigma(u)^{\pm 1}, \sigma(v)^{\pm 1}, \sigma(w)^{\pm 1})$$

is a symplectic automorphism if the parity of the number of ‘ -1 ’s agree with that of σ . Hence $\mathfrak{S}_4 = C_2^2 : \mathfrak{S}_3$ acts symplectically.

(a) and (b) generate a group isomorphic to C_2^3 . Hence the semi-direct product $H = C_2^3 : \mathfrak{S}_4$ acts on X symplectically. It is easily checked that this is H_{192} (cf. the remark on p. 192 in [18]). Since the involution $\varepsilon : (u, v, w) \mapsto (-u, -v, -w)$ commutes with the above automorphisms, H acts semi-symplectically on the quotient Enriques surface S . The actions (a) are Mathieu since all involutions have only elliptic curves as fixed curves. So is the action by (b) and the composite of (a) and (b) since they have only isolated fixed points. Hence the action (a)·(b) of C_2^3 and the automorphism $(u, v, w) \mapsto (v, w, u)$ generate a Mathieu action of $C_2 \times \mathfrak{A}_4$.

The automorphism

$$(3.1) \quad (u, v, w) \mapsto (\sqrt{-1}u, \sqrt{-1}v, -\sqrt{-1}/w)$$

is of order 4 and Mathieu. In fact, it has exactly four fixed points, two of which are symplectic and the rest of which are anti-symplectic. The automorphism (3.1) and the involution (a) generate a Mathieu action of $C_4 \times C_2$. \square

Remark 3.1. The Enriques surface of Example 6 is the normalization of the sextic surface $\sum_1^4 x_i^2 + \sqrt{-1}x_1x_2x_3x_4 \sum_1^4 x_i^{-2} = 0$ in \mathbb{P}^3 . The Mathieu actions of the two groups can be seen from this expression also. See [19, §6].

Remark 3.2. The complete intersection of three diagonal quadrics

$$\begin{cases} x_1^2 + x_3^2 + x_5^2 = x_2^2 + x_4^2 + x_6^2 \\ x_1^2 + x_4^2 = x_2^2 + x_5^2 = x_3^2 + x_6^2 \end{cases}$$

in \mathbb{P}^5 is given in [18, (0.4)] as a (smooth) $K3$ surface with a symplectic action of H_{192} . The automorphism $(x_i) \mapsto ((-1)^i x_i)$ is a free involution and commutes with the action. But the induced semi-symplectic action of $H_{192} = C_2^4 : D_{12}$ on the Enriques quotient is far from Mathieu. In fact, any (diagonal) involution in C_2^4 is not Mathieu. Hence any sub-action of $C_2 \times \mathfrak{A}_4$ or $C_4 \times C_2$ is not Mathieu neither.

4. SEMI-SYMPLECTIC AND MATHIEU AUTOMORPHISMS

Any Enriques surface is canonically doubly covered by a $K3$ surface. We always denote an Enriques surface by S and the $K3$ -cover by X . Let ω_X be a nowhere vanishing holomorphic 2-form on X . An automorphism φ is *symplectic* if it preserves the symplectic form ω_X . Equivalently, they are the elements in the kernel of the canonical representation $\text{Aut}(X) \rightarrow \text{GL}(H^0(\mathcal{O}_X(K_X)))$. Along the same line of ideas, we define the following.

Definition 4.1. Let S be an Enriques surface. An automorphism $\sigma \in \text{Aut}(S)$ is *semi-symplectic* if it acts trivially on the space $H^0(S, \mathcal{O}_S(2K_S))$.

The sections of $\mathcal{O}_S(2K_S)$ are identified with those of $\mathcal{O}_X(2K_X)$. Recall that the covering involution ε of X/S negates ω_X . Since for a given $\sigma \in \text{Aut}(S)$ we have two lifts φ_1 and $\varphi_2 = \varphi_1 \varepsilon$ to automorphisms of X , we get the following proposition.

Proposition 4.2. $\sigma \in \text{Aut}(S)$ is semi-symplectic if and only if one, say φ_1 , of the two lifts is symplectic. Moreover, the other lift $\varphi_2 = \varphi_1 \varepsilon$ negates ω_X .

Since we can uniformly choose the symplectic lift, we have also

Corollary 4.3. Let G be a group of semi-symplectic automorphisms of S . Then the lifts of automorphisms in G to the $K3$ -cover X constitute a group isomorphic to $G \times C_2$, where C_2 is generated by ε and whose subgroup $G \times \{\text{id}\}$ is the subgroup of symplectic automorphisms.

Example 4.4. Let S be a generic Enriques surface in the sense of [3]. Then the whole automorphism group $\text{Aut}(S)$ acts on S semi-symplectically. More generally, let S be an Enriques surface whose covering $K3$ surface X has the following genericity property:

- The transcendental lattice T_X of X , considered as a lattice equipped with Hodge structure, has only automorphisms $\{\pm \text{id}_{T_X}\}$.

Then the whole group $\text{Aut}(S)$ acts on S semi-symplectically. For example, this is the case when the Picard number $\rho(X)$ is odd, since the value of the Euler function $\phi(n)$ is an even number for all $n \geq 3$ and by [22].

4.1. Semi-symplectic automorphisms of finite order. From now on, we study automorphisms of Enriques surfaces of finite order, which we simply call finite automorphisms as a matter of convenience. First we have the following criterion.

Proposition 4.5. *Let σ be a finite automorphism of an Enriques surface S . If $\text{ord}(\sigma)$ is not divisible by 4, then it is automatically semi-symplectic.*

Proof. We separately argue two cases (1) $\text{ord}(\sigma) = 2$ and (2) $\text{ord}(\sigma)$ is odd. The rest is easily deduced.

(1) Let φ be one of two lifts to X . Then φ^2 is a lift of id_S hence is either id_X or ε . In the former case, either φ or $\varphi\varepsilon$ is symplectic on X and we are done by Proposition 4.2. Suppose the latter occurs. Then φ is an automorphism of order 4 on X which has no nontrivial stabilizer subgroup over X . Thus $X \rightarrow X/\varphi =: Y$ is an étale covering of degree 4. But then $\chi(\mathcal{O}_Y) = \chi(\mathcal{O}_X)/4 = 1/2$ must be an integer, a contradiction.

(2) Let n be the order of σ . Arguing as in (1), we see that the two lifts of σ have orders n and $2n$. We denote by φ the one with order n and we show that it is symplectic. Suppose that φ is non-symplectic. Then $Y := X/\varphi$ is a rational surface with at most quotient singularities. The free involution ε induces an involution ε_Y on Y , which must have a fixed point $P \in Y$ because $\chi(\mathcal{O}_Y) = 1$. Let $\pi_Y: X \rightarrow Y$ be the quotient morphism. Then the fiber $\pi_Y^{-1}(P)$ consists of odd number of points and has an action by ε by construction. However this is impossible since ε is fixed-point-free of order 2. Thus φ is symplectic. \square

Remark 4.6. The case (2) in the proof above also shows that if σ is an automorphism of odd order, then $S/\sigma \simeq X/\langle\varphi, \varepsilon\rangle$ is an Enriques surface (with singularities in general).

By [22, 18], a finite symplectic automorphism φ of a $K3$ surface has $\text{ord}(\varphi) \leq 8$ and the following table holds.

$$(4.1) \quad \begin{array}{c|cccccccc} & \text{order of } \varphi & & & & & & & \\ \hline & \text{number of fixed points} & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline & & 8 & 6 & 4 & 4 & 2 & 3 & 2 \end{array}$$

Corollary 4.7. *A finite semi-symplectic automorphism σ has $\text{ord}(\sigma) \leq 6$.*

Proof. By Corollary 4.3, σ has a symplectic lift φ of the same order, hence the order is at most 8. If $\text{ord}(\varphi) = 7$, then ε cannot act freely on the fixed point set $\text{Fix}(\varphi)$ by (4.1), a contradiction. If $\text{ord}(\varphi) = 8$, φ has exactly two fixed points P and Q and they are exchanged by ε . However, by applying the holomorphic Lefschetz formula, the local linearized actions $(d\varphi)_P, (d\varphi)_Q$ are given by

$$\begin{pmatrix} \zeta_8 & 0 \\ 0 & \zeta_8^7 \end{pmatrix} \text{ and } \begin{pmatrix} \zeta_8^3 & 0 \\ 0 & \zeta_8^5 \end{pmatrix} \quad \text{where } \zeta_8 = e^{2\pi\sqrt{-1}/8}.$$

These matrices are not conjugated by $d\varepsilon$ and we get a contradiction. Therefore we see that $\text{ord}(\sigma) \leq 6$. \square

Next we want to look more closely at the fixed point set $\text{Fix}(\sigma)$. Let φ be the symplectic lift and $\varphi\varepsilon$ be the non-symplectic one. From the relation $S = X/\varepsilon$, $\text{Fix}(\sigma)$ has the decomposition $\text{Fix}(\sigma) = \text{Fix}^+(\sigma) \sqcup \text{Fix}^-(\sigma)$, where $\text{Fix}^+(\sigma) = \text{Fix}(\varphi)/\varepsilon$ is the set of *symplectic* fixed points and $\text{Fix}^-(\sigma) =$

$\text{Fix}(\varphi\varepsilon)/\varepsilon$ is the set of *anti-symplectic* fixed points. Geometrically, they are distinguished by the determinant ($= \pm 1$) of the local linearized action $(d\sigma)_P$.

The number of symplectic fixed points is deduced from the table (4.1) by $\#\text{Fix}^+(\sigma) = \#\text{Fix}(\varphi)/2$. On the other hand, the anti-symplectic fixed point set $\text{Fix}^-(\sigma)$ has a variation. Here we use the topological Lefschetz formula

$$(4.2) \quad \sum_{i=0}^4 (-1)^i \text{tr}(\sigma^* |_{H^i(S, \mathbb{Q})}) = \chi_{\text{top}}(\text{Fix}(\sigma)).$$

to give a rough classification³. The quantity (4.2) is called the *Lefschetz number* and denoted by $L(\sigma)$.

Proposition 4.8. *Let σ be a semi-symplectic automorphism of order $n \leq 6$.*

- *If $n = 2$, then $\text{Fix}^-(\sigma)$ is a disjoint union of smooth curves. The Lefschetz number $L(\sigma)$ takes (every) even number from -4 to 12 .*
- *If $n = 3$, then $\text{Fix}^-(\sigma) = \emptyset$ and we have $L(\sigma) = 3$.*
- *If $n = 4$, then $\text{Fix}^-(\sigma)$ is either empty or 2 points. Accordingly $L(\sigma)$ equals either 2 or 4.*
- *If $n = 5$, then $\text{Fix}^-(\sigma) = \emptyset$ and we have $L(\sigma) = 2$.*
- *If $n = 6$, then $\text{Fix}^-(\sigma)$ is either empty or 2 points. Accordingly $L(\sigma)$ equals either 1 or 3.*

Proof. We have only to compute $\text{Fix}^-(\sigma)$ in each case.

($n = 2$) The first assertion follows since the local action $(d\sigma)_P$ at $P \in \text{Fix}^-(\sigma)$ is of the form $\text{diag}(1, -1)$. On the other hand, the action $\sigma^* \curvearrowright H^2(S, \mathbb{Q})$ is identified with $\varphi^* \curvearrowright H^2(X, \mathbb{Q})_{\varepsilon^*=1}$, the invariant subspace with respect to ε^* . It is known (also easily deduced from Table (4.1) via Lefschetz formula) that the action φ^* has an 8-dimensional negative eigenspace on $H^2(X, \mathbb{Q})$. Therefore, by counting the eigenvalues, we get the assertion (2). The existence of every value is shown in [13].

($n = 3, 5$) If the order n is odd, $((d\sigma)_P)^n \neq 1$ for $P \in \text{Fix}^-(\sigma)$. Therefore there are no anti-symplectic points.

($n = 4, 6$) Since $(\varphi\varepsilon)^2 = \varphi^2$, we have $\text{Fix}(\varphi\varepsilon) \subset \text{Fix}(\varphi^2)$. Since ε is free, we get $\text{Fix}(\varphi\varepsilon) \subset \text{Fix}(\varphi^2) - \text{Fix}(\varphi)$. This latter set T consists of 4 points in both cases $n = 4, 6$ and φ and ε both acts freely of order 2 on T . Thus we see that the fixed points of $\varphi\varepsilon$ are either whole T or empty. \square

4.2. Mathieu automorphisms. The left-hand-side of (4.2) can be regarded as the character of the representation $\sigma^* \curvearrowright H^*(S, \mathbb{Q})$ since odd dimensional (rational) cohomology vanishes. In contrast to symplectic automorphisms of $K3$ surfaces, Proposition 4.8 shows that we cannot treat semi-symplectic automorphisms uniformly from the viewpoint of characters. Nevertheless, we can make the following definition.

³For a detailed classification of involutions, see [13].

- Definition 4.9.** (1) A 12-dimensional representation V of a finite group G over a field of characteristic zero is called a *small Mathieu representation* if its character $\mu(g)$ depends only on $\text{ord}(g)$ and coincides with that of (the permutation representation on Ω_+ of) the small Mathieu group M_{11} .
- (2) Let G be a finite group of automorphisms of an Enriques surface S . The action is called *Mathieu* if it is semi-symplectic and the induced representation $G \curvearrowright H^*(S, \mathbb{Q})$ is a small Mathieu representation.

For the characters of M_{11} , see (**) in Introduction. We note that G acts on S effectively and without numerically trivial automorphisms if it is Mathieu. Comparing (**) and Proposition 4.8, we see that Mathieu condition has effects only on elements of even orders. A little stronger statement holds as follows.

Lemma 4.10. *A semi-symplectic group action of G on an Enriques surface S is Mathieu if for every element σ of order 2 or 4, we have $L(\sigma) = 4$.*

Proof. We prove that under the condition, any element $\sigma \in G$ of order 6 have $L(\sigma) = 1$. Let a_k ($k = 0, \dots, 5$) be the number of the eigenvalue $e^{2\pi k \sqrt{-1}/6}$ in the representation $\sigma^* \curvearrowright H^*(S, \mathbb{Q})$. Since $\chi_{\text{top}}(S) = 12$ and the representation is over \mathbb{Q} , we have

$$a_0 + \dots + a_5 = 12, \quad a_1 = a_5, \quad a_2 = a_4.$$

By assumption, we also have $L(\sigma^2) = 3$ and $L(\sigma^3) = 4$, which translates into

$$a_0 - a_1 - a_2 + a_3 = 3, \quad a_0 - 2a_1 + 2a_2 - a_3 = 4.$$

On the other hand, by Proposition 4.8 (6),

$$L(\sigma) = a_0 + a_1 - a_2 - a_3 = 1 \text{ or } 3.$$

The only integer solution to these equations is given by $a_0 = 4, a_1 = a_5 = 1, a_2 = a_4 = 2, a_3 = 2$. Therefore we get $L(\sigma) = 1$. \square

By Proposition 4.8 and Definition 4.9, we have

Proposition 4.11. *Let σ be a Mathieu automorphism of an Enriques surface of order $n \geq 2$. Then the fixed locus $\text{Fix}(\sigma) = \text{Fix}^+(\sigma) \sqcup \text{Fix}^-(\sigma)$ is as follows.*

n	2	3	4	5	6
$\text{Fix}^+(\sigma)$	4 pts.	3 pts.	2 pts.	2 pts.	1 pts.
$\text{Fix}^-(\sigma)$	$\sqcup_i C_i$	\emptyset	2 pts.	\emptyset	\emptyset

Here $\sqcup_i C_i$ is a disjoint union of smooth curves whose Euler number $\sum_i \chi_{\text{top}}(C_i)$ is zero.

5. FINITE GROUPS WITH SMALL MATHIEU REPRESENTATIONS

In this section we prove

Proposition 5.1. *Let G be a finite group which has a small Mathieu representation V with character μ . Then the order of G is*

$$2^{a_2} 3^{a_3} 5^{a_5} 11^{a_{11}}$$

for non-negative integers $a_2 \leq 4$, $a_3 \leq 2$, $a_5, a_{11} \leq 1$. Moreover, this bound is sharp since M_{11} has the order $7920 = 2^4 \cdot 3^2 \cdot 5 \cdot 11$.

The definition of a small Mathieu representation is in Definition 4.9. In what follows we use the notation $\mu(G) = (\sum_{g \in G} \mu(g))/|G|$. From the theory of characters, we have $\dim V^G = \mu(G)$. In particular,

$$(5.1) \quad \mu(G) \text{ is a non-negative integer.}$$

Note that a subgroup H inherits a small Mathieu representation and we have $\mu(H) \geq \mu(G)$. Also for a normal subgroup N , we can define the function μ on G/N and we have $\mu(G/N) \geq \mu(G)$. The equality holds if and only if $N = \{1\}$.

Proof. We begin the proof of Proposition 5.1. From Table (**), every element $g \in G$ has $\text{ord}(g) \in \{1, 2, 3, 4, 5, 6, 8, 11\}$. For a prime number p , let G_p be a Sylow p -subgroup of G . For odd p , G_p does not contain elements of order p^2 . Hence by (5.1),

$$\mu(G_p) = (12 + e_p(|G_p| - 1))/|G_p| \in \mathbb{Z} \quad (e_3 = 3, e_5 = 2, e_{11} = 1)$$

shows that $|G_3| \leq 3^2$, $|G_5| \leq 5$, $|G_{11}| \leq 11$.

In the rest, let us replace G by G_2 and show $|G| \leq 2^4$ for $p = 2$.

Lemma 5.2. *Abelian groups of order 2^4 have no small Mathieu representations.*

Proof. An abelian group of order 2^4 is isomorphic to either C_{16} , $C_8 \times C_2$, $C_4 \times C_4$, $C_4 \times C_2^2$ or C_2^4 . By definition C_{16} has no small Mathieu representation. For other groups we can easily compute $\mu(G)$ to see that they don't satisfy the condition (5.1). Hence we get the lemma. \square

Let A be a maximal normal abelian subgroup of $G = G_2$. We have $|A| \leq 2^3$ by Lemma 5.2. Since G is a 2-group, the centralizer $C_G(A)$ coincides with A and the natural homomorphism $\varphi: G/A \rightarrow \text{Aut}(A)$ is injective. Thus for $A \simeq C_2, C_4, C_2^2$, we have $\text{Aut}(A) \simeq \{1\}, C_2, \mathfrak{S}_3$ and we see that $|G| \leq 2^3$.

It remains to consider the case $|A| = 2^3$. There are three abelian groups $A \simeq C_8, C_4 \times C_2$ and C_2^3 . We treat them separately.

Case $A \simeq C_8$: Here we have $\text{Aut}(A) \simeq C_2^2$. To show $|G| \leq 2^4$, it suffices to show that φ is not surjective. Assume the contrary. Then there exists $x \in G$ which acts on the generator g of A by $xgx^{-1} = g^5$. We set $H = \langle A, x \rangle$. Since x^2 commutes with A , we get $x^2 \in A$ and $|H| = 2^4$. The equality

$(g^i x)^4 = g^{4i} x^4$ shows that $(g^i x)^4 = 1$ if $(x^4 = 1$ and i is even) or $(x^4 = g^4$ and i is odd), and $\text{ord}(g^i x) = 8$ otherwise. This enables us to compute

$$\mu(H) = \frac{1}{2}(\mu(A) + \mu(Ax)) = \frac{1}{2}(4 + 3) \notin \mathbb{Z}.$$

Therefore φ is not surjective.

Case $A \simeq C_4 \times C_2$: Let g, h be generators of A with $\text{ord}(g) = 4$ and $\text{ord}(h) = 2$. Then $\text{Aut}(A)$ is generated by

$$\begin{aligned} \alpha &: (g \ h) \mapsto (g + h \ 2g + h), \\ \beta &: (g \ h) \mapsto (g \ 2g + h), \end{aligned}$$

and is isomorphic to D_8 . Assume that $|\text{Im}(\varphi)| \geq 4$. Then G has an element x such that $\varphi(xA) = \alpha^2$ and $x^2 \in A$. By $(ax)^2 = x^2(a \in A)$, it follows that all elements in the coset Ax have the same order $\text{ord}(x) \in \{2, 4, 8\}$. Hence $\mu(Ax)$ is even, while $\mu(A) = 5$. Thus the group $H = \langle A, x \rangle$ cannot have a small Mathieu representation by (5.1), a contradiction. Hence $|\text{Im}(\varphi)| \leq 2$.

Case $A \simeq C_2^3$: We have $\text{Aut}(A) = \text{GL}(3, \mathbb{F}_2)$ which is the simple group of order 168. (One of) its 2-Sylow subgroups consist of elements

$$\begin{pmatrix} 1 & \alpha & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix}, \quad (\alpha, \beta, \gamma \in \mathbb{F}_2)$$

and is isomorphic to D_8 . Let us assume that the subgroup $\text{Im}(\varphi)$ has at least four elements. Then G contains an element x whose image by φ is conjugate to

$$B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^2$$

and $x^2 \in A$. In suitable coordinates of A , it is easy to see that every element in the coset Ax has order at most 4. Hence we get the contradiction

$$\mu(\langle A, x \rangle) = \frac{1}{2}(\mu(A) + \mu(Ax)) = \frac{1}{2}(5 + 4) \notin \mathbb{Z}.$$

Thus in all cases we get $|G| \leq 2^4$ and we obtain Proposition 5.1. \square

6. PROOFS OF THE MAIN THEOREM

We give the proofs to the main results, Theorems 2 and 7.

Theorem 6.1. *The following conditions are equivalent to each other for a finite group G .*

- (1) G has a Mathieu action on some Enriques surface.
- (2) G can be embedded into the symmetric group \mathfrak{S}_6 and the order $|G|$ is not divisible by 2^4 .
- (3) G is a subgroup of one of the following five maximal groups :

$$\mathfrak{A}_6, \mathfrak{S}_5, N_{72} = C_3^2 : D_8, C_2 \times \mathfrak{A}_4, C_2 \times C_4.$$

- (4) G is a group with a small Mathieu representation V with $\dim V^G \geq 3$, whose 2-Sylow subgroup is embeddable into \mathfrak{S}_6 and $G \not\cong Q_{12}$.

Remark 6.2. (1) There are 25 isomorphism classes of $G \neq \{1\}$ which satisfy the conditions (1)-(4) of the theorem. They are the groups exhibited in Propositions 6.8, 6.9 and 6.10.

(2) The group Q_{12} has a subtle behavior in the condition (4) of the theorem. Its 2-Sylow subgroup is obviously embedded into \mathfrak{S}_6 . Moreover, using the notation in [21, Table 2], the character $4\chi_0 + \chi_2 + 2\chi_3 + \chi_4 + \chi_5$ is a small Mathieu representation with $\dim V^{Q_{12}} = 4$. But it has no Mathieu actions by Lemma 6.4. It is thus necessary to put the extra condition on this group in the condition (4).

We have already shown (3) \Rightarrow (1) in Section 3. In the following three subsections we prove the rest in the order (1) \Rightarrow (4) \Rightarrow (2) \Rightarrow (3).

6.1. Proof of (1) \Rightarrow (4). By definition, $H^*(S, \mathbb{Q})$ is a small Mathieu representation. Obviously $H^i(S, \mathbb{Q})$ ($i = 0, 4$) are invariant subspaces and for any ample divisor H on S the sum $H' = \sum_{g \in G} gH$ is a G -invariant ample divisor, showing $H^2(S, \mathbb{Q})^G \neq 0$. Putting together, we find $\dim H^*(S, \mathbb{Q})^G \geq 3$.

Next we show that the 2-Sylow subgroup G_2 is embeddable in \mathfrak{S}_6 . By Corollary 4.7 every element $g \in G_2$ has $\text{ord}(g) \leq 4$. By the character table (**), we see that $\mu(g) = 4$ unless $g = 1$. Thus the condition (5.1),

$$\mu(G_2) = \frac{1}{|G_2|}(12 + 4(|G_2| - 1)) \in \mathbb{Z}$$

gives $|G_2| \leq 2^3$. It is easy to check that all 2-groups of order at most 8, except for $G_2 = C_8$ and Q_8 , can be embedded in the group $C_2 \times D_8$, the 2-Sylow subgroup of \mathfrak{S}_6 . The cyclic group C_8 is impossible by Corollary 4.7. The group Q_8 is also impossible by the following lemma, which concludes $G_2 \subset \mathfrak{S}_6$.

Lemma 6.3. *No Mathieu actions on Enriques surfaces by the quaternion group, $Q_8 = \langle g, h \mid g^4 = 1, hgh^{-1} = g^{-1}, g^2 = h^2 \rangle$, exist.*

Proof. By means of contradiction, suppose that we had one. We denote by $i = g^2$ the unique and central involution in Q_8 . By Proposition 4.11, we see that the fixed point sets of g and h both coincide with the set of four isolated fixed points of i . In particular, these four points are fixed by the whole group.

Let P be one of anti-symplectic fixed points of g , which exists by Proposition 4.11. By looking at differentials, we obtain a map $d_P: Q_8 \rightarrow \text{GL}(T_P S)$, which is injective by the complete reducibility for finite groups. But since any embedding of Q_8 into $\text{GL}(2, \mathbb{C})$ factors through $\text{SL}(2, \mathbb{C})$, this contradicts to that P was an anti-symplectic fixed point. This proves the lemma. (Proof of the latter fact: Note that the diagonal form of the involution $d_P(i)$ is either $\text{diag}(1, -1)$ or $\text{diag}(-1, -1)$. In the former case, its centralizer in $\text{GL}(2, \mathbb{C})$ is the commutative group of diagonal matrices, hence we get a

contradiction. In the latter case, from $g^2 = h^2 = i$ and $gh \neq hg$, we must have that both $d_P(g)$ and $d_P(h)$ have $\text{tr} = 0$ and $\det = 1$. \square

Finally we show $G \not\cong Q_{12}$.

Lemma 6.4. *No Mathieu actions on Enriques surfaces by the group $Q_{12} = \langle g, h \mid g^6 = 1, h^2 = g^3, hgh^{-1} = g^{-1} \rangle$ exist.*

Proof. Assume we had one. Since g^3 is the unique involution in Q_{12} , by Proposition 4.11, we see that the fixed point sets of h and gh both coincide with the set of four isolated fixed points of g^3 . We denote them by P_i ($i = 1, \dots, 4$). However from the equations $h(P_i) = P_i, gh(P_i) = P_i$ we get $g(P_i) = P_i$, contradicting to that g of order 6 has a unique isolated fixed point by Proposition 4.11. \square

6.2. Proof of (4) \Rightarrow (2).

Lemma 6.5. *Let G have a small Mathieu representation and assume that $\mu(G) \geq 3$. Then 11 does not divide $|G|$.*

Proof. If G has a nontrivial 11-Sylow subgroup G_{11} , the dimensions of invariant subspaces satisfy $2 = \mu(G_{11}) \geq \mu(G)$, a contradiction. \square

Lemma 6.6. *Let G have a small Mathieu representation and assume that G_2 is embeddable into \mathfrak{S}_6 . Then G has no elements of order 8 and $|G_2| \leq 2^3$.*

Proof. Since the 2-Sylow subgroup $(\mathfrak{S}_6)_2$ is isomorphic to $C_2 \times D_8$, G_2 has no elements of order 8. By the definition of small Mathieu character μ , every non-identity element $g \in G_2$ has character $\mu(g) = 4$. Thus the condition

$$\mu(G_2) = \frac{1}{|G_2|}(12 + 4(|G_2| - 1)) \in \mathbb{Z}$$

gives $|G_2| \leq 8$. \square

Corollary 6.7. *Let G be a finite group that satisfies the condition (4) of Theorem 6.1. Then for all $g \in G$ we have $\text{ord}(g) \leq 6$. Moreover we have*

$$|G| = 2^{a_2} 3^{a_3} 5^{a_5}$$

for non-negative integers $a_2 \leq 3, a_3 \leq 2, a_5 \leq 1$.

Proof. This follows immediately by combining Proposition 5.1 and lemmas above. \square

In particular, we have proved the latter part of (2) of Theorem 6.1.

In the following, we classify all groups that satisfy the condition (4). First we consider non-solvable groups. Recall that G is non-solvable if and only if at least one of its composition factors is a non-abelian finite simple group.

Proposition 6.8. *Let G be a finite group that satisfies the condition (4) of Theorem 6.1. Assume that G is non-solvable. Then G is isomorphic either to $\mathfrak{A}_5, \mathfrak{S}_5$ or \mathfrak{A}_6 .*

Proof. Let N be a composition factor of G which is a non-abelian simple group. By Corollary 6.7 and the table of finite simple groups (see [2]), N is either \mathfrak{A}_5 or \mathfrak{A}_6 . In the latter case we see $G = N \simeq \mathfrak{A}_6$ by Corollary 6.7.

Let us continue the case $N \simeq \mathfrak{A}_5$. By order reason, N is the only non-abelian simple factor. We have subgroups $H \subset G$ and $T \triangleleft H$ such that $H/T \simeq \mathfrak{A}_5$. Since H inherits the small Mathieu representation and \mathfrak{A}_5 is a quotient of H , we have

$$3 = \mu(\mathfrak{A}_5) = \mu(H/T) \geq \mu(H) \geq \mu(G)$$

by the discussion after Proposition 5.1. By the condition $\dim V^G \geq 3$, we have equalities. It follows that T is trivial and $H \simeq \mathfrak{A}_5$.

By Corollary 6.7, the index $[G : H]$ is a divisor of 6. Hence the composition series of G looks either (i) $H \triangleleft G$ or (ii) $H \triangleleft G' \triangleleft G$ (if it has more than one terms). Let us begin with (i). We consider the natural homomorphism $\varphi: G \rightarrow \text{Aut}(H)$. Since G does not contain elements of order $5n$ ($n \geq 2$) and H is center-free, we see that φ is injective into $\text{Aut}(\mathfrak{A}_5) \simeq \mathfrak{S}_5$. Therefore G is isomorphic to \mathfrak{S}_5 . In the case (ii), we get $G' \simeq \mathfrak{S}_5$ by (i). We again consider the natural homomorphism $\psi: G \rightarrow \text{Aut}(G')$. By the same reasoning as before, ψ is injective. But this is not the case since $G \supset G' \simeq \mathfrak{S}_5$ is a proper inclusion.

Thus we obtain the classification of non-solvable groups. \square

We recall that G is nilpotent if and only if G is the direct product of its Sylow subgroups.

Proposition 6.9. *Let G be a nilpotent group that satisfies the condition (4) of Theorem 6.1. Then $G \simeq C_n^a$ ($2 \leq n \leq 6$ if $a = 1$ and otherwise $(n, a) = (2, 2), (3, 2), (2, 3)$), $C_2 \times C_4$ or D_8 .*

Proof. Corollary 6.7 and the given condition on 2-Sylow subgroups classify the Sylow subgroups of G as follows: G_2 is isomorphic to $C_2^3, C_2 \times C_4, D_8$ or has order at most 4, G_3 is isomorphic to C_3^2 or C_3 and G_5 is isomorphic to C_5 (if not trivial).

We claim that neither groups $H_1 = C_2 \times C_3^2$ nor $H_2 = C_2^2 \times C_3$ have small Mathieu representations. In fact, the former has $\mu(H_1) = 8/3 \notin \mathbb{Z}$. For the latter, let us choose generators $g, h \in H_2$ with $g^6 = h^2 = ghg^{-1}h^{-1} = 1$. Let ψ be the character of H_2 assigning $h \mapsto 1$ and $g \mapsto \zeta_6$, where ζ_6 is the primitive 6-th root of unity. Then the inner product of characters (ψ, μ) is $1/2 \notin \mathbb{Z}$. Thus H_2 does not have small Mathieu representations.

Recall that all elements $g \in G$ have $\text{ord}(g) \leq 6$ by Corollary 6.7. This fact with the non-existence of subgroups H_i above leads us to the list. \square

Finally we treat the case G is non-nilpotent and solvable. Recall that any finite group has the maximal normal nilpotent subgroup F , called the *Fitting subgroup*. When G is non-nilpotent and solvable, F is a proper subgroup and it is known that the centralizer $C_G(F)$ coincides with the center $Z(F)$ of F . In particular the natural homomorphism $\varphi: G/F \rightarrow \text{Out}(F)$ is injective,

where $\text{Out}(F) = \text{Aut}(F)/\text{Inn}(F)$ is the group of outer automorphism classes. Moreover, by the extended Sylow's theorem for solvable groups, the exact sequence

$$(6.1) \quad 1 \longrightarrow F \longrightarrow G \longrightarrow G/F \longrightarrow 1$$

splits if $|F|$ and $|G/F|$ are coprime.

Proposition 6.10. *Let G be a non-nilpotent and solvable group that satisfies the condition (4) of Theorem 6.1. Then G belongs to the following list.*

G	D_6	D_{10}	D_{12}	\mathfrak{A}_4	$\mathfrak{A}_{3,3}$	$C_3 \times \mathfrak{S}_3$
$ G $	6	10	12	12	18	18

G	$\text{Hol}(C_5)$	$C_2 \times \mathfrak{A}_4$	\mathfrak{S}_4	$C_3^2 \rtimes C_4$	$\mathfrak{S}_{3,3}$	N_{72}
$ G $	20	24	24	36	36	72

Proof. Let F be the Fitting subgroup. The isomorphism class of F belongs to the list of Proposition 6.9, so we give separate considerations. We note that $\text{Out}(F) = \text{Aut}(F)$ if F is abelian.

Case: F is cyclic In the table below, -1 denotes the inversion $g \mapsto g^{-1}$ ($g \in F$).

F	C_2	C_3	C_4	C_5	C_6
$\text{Aut}(F)$	$\{1\}$	$\{\pm 1\}$	$\{\pm 1\}$	C_4	$\{\pm 1\}$
G		D_6		$D_{10}, \text{Hol}(C_5)$	D_{12}

For $F = C_2, C_4$ all extensions (6.1) are nilpotent by order reasoning. For $F = C_3, C_5$, (6.1) splits and we get the table. Here $\text{Hol}(C_5)$ denotes the holomorph $C_5 \rtimes \text{Aut}(C_5)$. For $F = C_6$, we get a split extension D_{12} . the other non-split extension Q_{12} is not allowed by the assumption.

Case: $F = C_2^2$ We have $\text{Aut}(F) = \mathfrak{S}_3$. Since G is non-nilpotent, it has an element x of order 3. Then F and x generate a subgroup H isomorphic to \mathfrak{A}_4 . If further the inclusion $H \subset G$ is proper, H has index two and is normal. In this case, such G is isomorphic to \mathfrak{S}_4 or $C_2 \times \mathfrak{A}_4$, but in the latter group C_2^2 is not the Fitting subgroup.

Case: $F = C_3^2$ We have $\text{Aut}(F) \simeq \text{GL}(2, \mathbb{F}_3)$ which has order $48 = 2^4 \cdot 3$. This group has the semi-dihedral group SD_{16} as its 2-Sylow subgroup,

$$SD_{16} = \langle g, x \mid g^8 = x^2 = 1, xgx^{-1} = g^3 \rangle.$$

By Corollary 6.7, the extension (6.1) splits and G/F is a subgroup of SD_{16} . Since the maximal subgroups of SD_{16} are C_8, D_8, Q_8 and all are characteristic, we see that isomorphic subgroups of order 8 in $\text{GL}(2, \mathbb{F}_3)$ are conjugate. In view of Proposition 6.9, we get the unique extension $G \simeq C_3^2 \rtimes D_8 = N_{72}$ if $|G|$ is maximal. If $|G/F| = 4$, we get unique extensions $C_3^2 \rtimes C_4, C_3^2 \rtimes C_2^2 \simeq \mathfrak{S}_{3,3}$. If $|G/F| = 2$, we have two extensions which are isomorphic to $\mathfrak{A}_{3,3}, C_3 \times \mathfrak{S}_3$.

Case: $F = C_2^3$ We have $\text{Aut}(F) = \text{GL}(3, \mathbb{F}_2)$, which is the simple group of order $168 = 2^3 \cdot 3 \cdot 7$. By Corollary 6.7, (6.1) splits with $G/F \simeq C_3$. Since 3-Sylow subgroups in $\text{GL}(3, \mathbb{F}_2)$ are conjugate, we get the unique extension $G \simeq C_2 \times \mathfrak{A}_4$.

Case: $F = C_2 \times C_4, D_8$ In these cases we have $\text{Aut}(F) \simeq D_8$, hence we get no non-nilpotent groups.

This completes the classification of possible groups. It is not difficult to check that every group is in \mathfrak{S}_6 , and we have proved our theorem. \square

6.3. Proof of (2) \Rightarrow (3). By condition, $G \neq \mathfrak{S}_6$. By [2], \mathfrak{S}_6 has four isomorphism classes of maximal subgroups $\mathfrak{A}_6, \mathfrak{S}_5, N_{72}, C_2 \times \mathfrak{S}_4$, the first three of which readily satisfy (3). Thus we may assume $G \subset C_2 \times \mathfrak{S}_4$. Again by the order condition G is a proper subgroup. A standard argument shows that $C_2 \times \mathfrak{S}_4$ has maximal subgroups $\mathfrak{S}_4, C_2 \times \mathfrak{S}_3, C_2 \times \mathfrak{A}_4, C_2 \times D_8$. The first two are subgroups of \mathfrak{S}_5 . The third one is in the list of (3). Thus we may assume $G \subset C_2 \times D_8$. Again by the order condition G is a proper subgroup, and the maximal subgroups of $C_2 \times D_8$ are $C_2^3, D_8, C_2 \times C_4$. The first two groups are subgroups of $C_2 \times \mathfrak{A}_4$ and N_{72} respectively. The last group $C_2 \times C_4$ is the final ingredient in (3), so our assertion holds.

7. TAME AUTOMORPHISMS IN POSITIVE CHARACTERISTIC

Let k be an algebraically closed field of positive characteristic $p > 0$. Recall that an Enriques surface S over k is characterized by the numerical equivalence $K_S \equiv 0$ and the second Betti number $b_2(S) = 10$, including $p = 2$. Let G be a finite group of tame automorphisms acting on S . Since $\dim H^0(S, \mathcal{O}_S(2K_S)) = 1$ and $\dim H^*(S, \mathbb{Q}_l) = 12$ ($l \neq p$), the same definitions as Definition 4.1 and 4.9 (2) make sense. Hence we can speak of Mathieu automorphisms over k as well. In this section we use a result of Serre [24] to prove Theorem 8. First we remark the following.

Proposition 7.1. *In characteristic $p \geq 11$, any semi-symplectic finite group action on an Enriques surface is tame.*

Proof. Assume $p = \text{char } k \geq 11$ and σ is an automorphism of order p of an Enriques surface S over k . Then it lifts to the canonical $K3$ -cover X and commutes with the covering involution ε . The lift, which we denote by $\tilde{\sigma}$, is a wild automorphism of a $K3$ surface, hence by [11, Theorem 2.1], $p = 11$ follows. Moreover, by [12, Lemma 2.3], $\dim H_{\text{et}}^2(X, \mathbb{Q}_l)^\sigma = 2$ for all $l \neq 11$. By choosing a prime l so that the cyclotomic polynomial $T^{10} + \dots + 1$ is irreducible over \mathbb{Q}_l , this shows that the second cohomology is a sum of four irreducible modules of dimension 1, 1, 10, 10 over \mathbb{Q}_l . ($l = 2$ suffices, for example.) But since $H_{\text{et}}^2(X, \mathbb{Q}_l)^\varepsilon$ is 10 dimensional and contains a $\tilde{\sigma}$ -invariant class, $\tilde{\sigma}$ and ε cannot commute. Therefore we obtain a contradiction. \square

More explicitly Theorem 8 is stated as follows.

Theorem 7.2. *A finite group G has a tame Mathieu action on some Enriques surface S over k if and only if*

- (1) G satisfies the same conditions as (1)-(4) of Theorems 2 and 7 when $\text{char } k \geq 7$.

- (2) G is isomorphic to a subgroup of N_{72} , \mathfrak{S}_4 , $C_2 \times \mathfrak{A}_4$ or $C_2 \times C_4$ when $\text{char } k = 5$.
- (3) G is isomorphic to a subgroup of $\text{Hol}(C_5)$, D_8 , C_2^3 or $C_2 \times C_4$ when $\text{char } k = 3$.
- (4) G is isomorphic to a subgroup of C_3^2 or C_5 when $\text{char } k = 2$.

7.1. Theorem 7.2 in odd characteristics. First we prove the ‘only if’ part of Theorem 7.2. Serre’s theorem [24, Theoreme 5.1] says that if X is a smooth projective variety over k with a tame automorphism group G with $H^2(X, \mathcal{O}_X) = H^2(X, \Theta_X) = 0$, then (X, G) lifts to characteristic zero. If $\text{char } k \geq 3$ and S is an Enriques surface over k , the assumptions are satisfied and hence (S, G) lifts smoothly to characteristic zero. By our result over \mathbb{C} , we see that G is one of the 25 groups of Theorems 2 and 7. By the condition of tameness, we can deduce the classification of groups from Section 6. This gives the proof of ‘only if’ part of Theorem 7.2 for odd characteristics.

Now we discuss the reductions modulo p of our maximal group actions.

- (1) Example 3 degenerates in characteristic $p = 2, 5$ since ε becomes to have a fixed point $(1 : 1 : 1 : 1 : 1) \in \mathbb{P}^4$. In other characteristics, the same equation defines an Enriques surface and the group \mathfrak{S}_5 acts in the same way.
- (2) The surface (2.1) with $\lambda, \mu = 1 \pm \sqrt{3}$ becomes reducible in characteristics $p = 2, 3$. In fact, it contains the 2-plane $x_i = y_i$, $i = 0, 1, 2$. In other characteristics, we can check that the surface X (with $\lambda, \mu = 1 \pm \sqrt{3}$) is smooth and ε has no fixed points. Moreover, the studies in Subsections 2.1 and 2.2 both hold true without changes. Therefore the Mathieu actions by N_{72} and \mathfrak{A}_6 exist in all characteristics $p \geq 5$.
- (3) The equation of Example 6 becomes reducible in $p = 2$, but in other characteristics $p \geq 3$ the surface is smooth and the group action by H_{192} remains well-defined. Therefore we also have the Mathieu actions by $C_2 \times C_4$ and $C_2 \times \mathfrak{A}_4$ in characteristic $p \geq 3$.

Now we prove the ‘if’ part. If $\text{char } k \geq 7$, then all the Examples 3, 4, 5 and 6 persist as Enriques surface. Therefore all groups satisfying the conditions of Theorem 2 and 7 have Mathieu actions on some Enriques surface over k . In $\text{char } k = 5$, since \mathfrak{S}_4 is a subgroup of \mathfrak{A}_6 , our result follows from the above study on Examples 4, 5 and 6. In $\text{char } k = 3$, since $\text{Hol}(C_5)$ and D_8 are subgroups of \mathfrak{S}_5 and C_2^3 is contained in $C_2 \times \mathfrak{A}_4$, our result follows from the above study on Examples 3 and 6. This finishes the proof of Theorem 7.2 for odd characteristics.

7.2. Theorem 7.2 in characteristic 2. Finally we show Theorem 7.2 (4). Since the conditions of Serre’s theorem are often invalid, we give a direct treatment of the ‘only if’ part modifying the proof of [23, Proposition 2].

Let $\pi: S \rightarrow Y = S/\sigma$ be the quotient morphism by a tame semi-symplectic automorphism σ . Let $P \in S$ be a fixed point of σ . By the tameness, the action is locally linearizable at P and we have the complete reducibility on $\mathcal{O}_{S,P}$ as in [18, (1.1)]. Therefore, $\pi(P)$ is a rational double point and Y is smooth except those isolated singularities. Moreover, there is a nowhere vanishing global bi-canonical form on Y , since σ is semi-symplectic. In particular, we have $K_{\tilde{Y}} \equiv 0$ for the minimal resolution \tilde{Y} of Y . Hence we have

$$(7.1) \quad c_2(\tilde{Y}) = 12\chi(\mathcal{O}_{\tilde{Y}}) \leq 24$$

by Noether's formula.

Lemma 7.3. *Let σ be a tame semi-symplectic automorphism in characteristic 2 with order q , an odd prime number. Then one of the following holds.*

- (1) σ is of order 3 and has three fixed points on S .
- (2) σ is of order 5 and has two fixed points on S .

In particular, σ is automatically of Mathieu type.

Proof. Let $\pi: S \rightarrow Y = S/\sigma$ and $P \in \text{Fix } \sigma$ be as above. Then $\pi(P)$ is a rational double point of type A_{q-1} . Denoting the number of fixed points by r , we have

$$(7.2) \quad 12 = c_2(S) = q(c_2(\tilde{Y}) - qr) + r$$

The integer solution of (7.2) exists in the range (7.1) only when \tilde{Y} is an Enriques surface and $(q, r) = (3, 3), (5, 2), (11, 1)$. But if $q = 11$, then the exceptional curves of resolution of the singularity span the negative definite lattice A_{10} of rank 10 in $NS(\tilde{Y})$, a contradiction to $\rho(\tilde{Y}) = 10$. This shows the assertions (1) and (2). The last statement follows from the Lefschetz fixed point formula. \square

Lemma 7.4. *There are no automorphisms of order 9, 15 or 25. Therefore 3 and 5 are the only possible orders.*

Proof. If σ is of order 9, then $\text{Fix}(\sigma)$ is either empty or coincides with $\text{Fix}(\sigma^3)$. In the former case the equation corresponding to (7.2) is

$$12 = 9(c_2(\tilde{Y}) - 3) + 3$$

and $c_2(\tilde{Y}) = 4$, which is impossible. In the latter case Y has three rational double points of type A_8 , whose rank is too large. Hence automorphisms of order 9 do not exist. The other cases are treated in the same way. \square

Now we are ready to prove the ‘only if’ part. Assume that a finite group G has a tame action. By Lemma 7.3 and 7.4, G is of Mathieu type. The discussion in Subsection 6.2 is purely group-theoretic and remains true in our setting, too. Hence, we get the three groups by the lists in Propositions 6.8, 6.9 and 6.10.

Finally we show the existence of actions of C_3^2 and C_5 .

- (1) Let $X = X_{\lambda, \mu}$ be the same as (2.1) but we assume further that both $\lambda, \mu \in k$ are nonzero in characteristic 2. Then X has 12 nodes at the intersection of 12 conics defined in Section 2, for example at $(1 : 0 : 0 : 1 : 0 : 0)$ and $(1 : 1 : 1 : \alpha : \alpha : \alpha)$, etc., with $\alpha = \sqrt{(1-\lambda)/(1-\mu)}$. Moreover, X is smooth elsewhere. These nodes correspond to the 12 double points in fibers of the rational elliptic surface (2.3). The usual involution $\varepsilon: (x : y) \mapsto (x : -y)$ of \mathbb{P}^5 in characteristic $\neq 2$ is replaced by the action of the non-reduced group scheme $\mu_2 = \text{Spec } k[t]/(t^2 - 1)$ defined by $(x : y) \mapsto (x : ty)$. A local computation shows that 12 nodes disappear and the quotient X/μ_2 becomes a smooth (classical) Enriques surface. The group C_3^2 acts on X/μ_2 by (2.2).
- (2) Let X be the surface defined by the same equation as (1.2) in characteristic 2. It has only 10 nodes and by taking the minimal resolution of the quotient by the Cremona involution, we obtain a smooth (non-classical) Enriques surface with \mathfrak{S}_5 -action. In particular, it has a C_5 action.

Thus the proof of Theorem 7.2(=Theorem 8) is completed.

APPENDIX A. LATTICE THEORETIC CONSTRUCTION OF MATHIEU ACTIONS

We give a lattice theoretic proof of Theorem 1.2.

In [17, Appendix], for each G of the eleven groups $(*)$, a symplectic action on a K3 surface is constructed using

- (1) the Niemeier lattice N of type $(A_1)^{24}$,
- (2) the action of the Mathieu group M_{24} on N ,
- (3) an embedding of G into the Mathieu group M_{23} , and
- (4) the Torelli type theorem for K3 surfaces.

Here N is even, unimodular and contains the root lattice

$$(A.1) \quad \bigoplus_{i \in \Omega} \mathbb{Z}e_i, \quad (e_i^2) = -2$$

as a sublattice of finite ($= 2^{12}$) index, where Ω is the operator domain of M_{24} . The action $M_{24} \curvearrowright \Omega$ extends (isometrically) on N . The key of the proof is to show the existence of a primitive embedding of N_G in the K3 lattice $\Lambda \simeq 3U + 2E_8$, where N_G is the orthogonal complement of the invariant lattice $N^G \subset N$.

In this appendix, making this construction C_2 -equivariant, we give another proof of Theorem 1.2. Namely we decompose (A.1) in two parts

$$(A.2) \quad \bigoplus_{i \in \Omega_+} \mathbb{Z}e_i \quad \text{and} \quad \bigoplus_{i \in \Omega_-} \mathbb{Z}e_i, \quad (e_i^2) = -1, \quad \forall i \in \Omega = \Omega_+ \sqcup \Omega_-$$

with scaling by $1/2$. Let N_{\pm} be the lattices obtained by adding $(\sum_{i \in \Omega_{\pm}} e_i)/2$ to these. N_{\pm} is the dual of the root lattice of type D_{12} , and $N_{\pm}(2)$ is an integral lattice of discriminant 2^{10} .

Let $G_{(6)}, G_{(9)}, G_{(10)} \subset M_{11}$ be the image of the embedding of the three groups $\mathfrak{S}_5, N_{72}, \mathfrak{A}_6$ in Lemma 1.1, respectively. $G_{(n)}$ decomposes the operator domain $\Omega_+ \setminus \{\star\}$ of M_{11} into two orbits of length n and $11 - n$. Let Ω_- be the complementary dodecad of Ω_+ . The following is immediate from the proof of Lemma 1.1.

Lemma A.1. *Each $G_{(n)}$ decomposes Ω_- into two orbits. Their length are $\{6, 6\}$ when $n = 9, 10$, and $\{2, 10\}$ when $n = 6$.*

We consider the orthogonal complement of the invariant lattice for two actions $G_{(n)} \curvearrowright N_{\pm}$. The following is obvious.

Lemma A.2. *Let $G = G_{(n)} \curvearrowright N_{\pm}$ ($n = 6, 9, 10$) be as above.*

(1) *The orthogonal complement $N_{+,G}$ of the invariant lattice $N_+^G \subset N_+$ is the root lattice of type $A_{n-1} + A_{10-n}$.*

(2) *The orthogonal complement $N_{-,G}$ of the invariant lattice $N_-^G \subset N_-$ is a negative definite odd integral lattice of rank 10. $N_{-,G}$ contains an index-two sublattice which is of type $A_5 + A_5$ when $n = 9, 10$ and $A_1 + A_9$ when $n = 6$.*

The Niemeier lattice N contains $N_{\pm}(2)$ as a primitive sublattice. Since N is unimodular we have an isomorphism

$$(A.3) \quad \text{Disc } N_+(2) \simeq \text{Disc } N_-(2)$$

of discriminant groups. This isomorphism is compatible with the actions of M_{12} .

Now we recall the modulo 2 reduction $l := L/2L$ of an integral lattice L . When L is even, l is endowed with the quadratic form

$$q : l \rightarrow \mathbb{Z}/2\mathbb{Z}, \quad x \mapsto (x^2)/2$$

with value in $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$. When L is odd, l is endowed with the bilinear form

$$(A.4) \quad b : l \times l \rightarrow \left(\frac{1}{2}\mathbb{Z}\right)/\mathbb{Z}, \quad (x, y) \mapsto (x \cdot y)/2.$$

The alternating part $l^{alt} := \{x \mid b(x, x) = 0\}$ of l is a subspace of codimension one, and carries

$$(A.5) \quad q : l^{alt} \rightarrow \mathbb{Z}/2\mathbb{Z}, \quad x \mapsto (x^2)/2$$

which is a *quadratic refinement* of b , that is, $q(x + y) - q(x) - q(y) = b(x, y)$ holds for every $x, y \in l^{alt}$.

We need an Enriques counterpart of the isomorphism (A.3). The $K3$ lattice Λ decomposes in two parts by the action of free involution ε . The invariant part is the Enriques lattice Λ_+ of type $T_{2,3,7}$ scaled by 2. The anti-invariant part Λ_- , called the anti-Enriques lattice, is isomorphic to

$U + U(2) + E_8(2)$. Since Λ and Λ_+ are unimodular and since Λ contains the orthogonal direct sum $\Lambda_+(2) + \Lambda_-$, we have the isomorphism

$$(A.6) \quad \Lambda_+/2\Lambda_+ \simeq \text{Disc } \Lambda_-$$

of 10-dimensional quadratic spaces over \mathbb{F}_2 .

Returning to the action $G = G_{(n)} \curvearrowright N_{\pm}$, we put $L_{\pm} := N_{\pm, G}$ and denote its modulo 2 reduction by l_{\pm} . Restricting the isomorphism (A.3) to l_+ we have

Lemma A.3. *Two 9-dimensional quadratic spaces (l_+, q_+) and (l_-^{alt}, q_-) over \mathbb{F}_2 are isomorphic to each other including their G -actions.*

Remark A.4. The bilinear form on (l_+, q_+) has 1-dimensional radical, and q_+ takes value 1 at the nonzero element in the radical.

The lattice N_G , the orthogonal complement of $N^G \subset N$, is obtained by patching two lattices L_{\pm} by the isomorphism in the lemma. N_G is an even lattice of *Leech type*, that is, the induced action of G on the discriminant group $\text{Disc } N_G$ is trivial and N_G does not have a (-2) element.

As is observed in the introduction L_+ have a primitive embedding into the lattice of type $T_{2,3,7}$. The following is the counterpart of L_- .

Proposition A.5. *The lattice $L_-(2)$ has a primitive embedding into the anti-Enriques lattice Λ_- .*

The essential part is this.

Lemma A.6. *L_- has a primitive embedding into the odd unimodular lattice $I_{2,10} := \langle 1 \rangle^2 + \langle -1 \rangle^{10}$ of signature $(2, 10)$.*

Proof. We take $\{h_1, h_2, e_1, \dots, e_{10}\}$ with $(h_1^2) = (h_2^2) = 1$ and $(e_i^2) = -1$ ($1 \leq i \leq 10$) as an orthogonal basis of $I_{2,10}$.

In the case $n = 6$, L_- is the unique (odd) integral lattice containing $A_1 + A_9$ as a sublattice of index 2. $e_i - e_{i+1}$ ($i = 1, \dots, 9$) and $v = 2(h_1 + h_2) - \sum_{i=1}^{10} e_i$ generate a root sublattice of type $A_9 + A_1$ in $I_{2,10}$. Since the half sum of $e_{2j-1} - e_{2j} \in A_9$ ($j = 1, \dots, 5$) and v belongs to $I_{2,10}$, the primitive hull of $A_9 + A_1 \subset I_{2,10}$ is isomorphic to L_- .

In the case $n = 9, 10$, L_- is the unique integral lattice containing $A_5 + A_5$ as a sublattice of index 2. L_- is isomorphic to the orthogonal complement of $2(h_1 + h_2) - e_1 - e_2 - e_3 - e_4 - e_5$ and $2(h_1 - h_2) - e_6 - e_7 - e_8 - e_9 - e_{10}$ in $I_{2,10}$. In fact, the orthogonal complement is generated by

$$\begin{aligned} &e_1 - e_2, e_2 - e_3, e_3 - e_4, e_4 - e_5, h_1 + h_2 - e_1 - e_2 - e_3 - e_4; \\ &e_6 - e_7, e_7 - e_8, e_8 - e_9, e_9 - e_{10}, h_1 - h_2 - e_6 - e_7 - e_8 - e_9 \end{aligned}$$

which form a root lattice R of type $A_5 + A_5$, and $h_1 - e_2 - e_4 - e_6 - e_8 \notin R$. \square

Remark A.7. In the above proof we make use of the fact that the blow-up of the projective space \mathbb{P}^3 at five points has a Cremona symmetry of type A_5 , which is described by Dolgachev[10] in terms of root systems.

Proof of Proposition A.5. The anti-Enriques lattice Λ_- is obtained from $I_{2,10}(2)$ by adding (-2) -element $(h_1 + h_2 - \sum_1^{10} e_i)/2$. Since $h_1 + h_2 - \sum_1^{10} e_i$ does not belong to the modulo 2 reduction of the image of $L_- \hookrightarrow I_{2,10}$ constructed in the lemma, the induced embedding $L_-(2) \hookrightarrow \Lambda_-$ is also primitive. \square

Remark A.8. The above relation between the anti-Enriques lattice Λ_- and $I_{2,10}$ is observed in Allcock[1].

Patching together two primitive embeddings $L_+(2) \hookrightarrow \Lambda_+(2)$, determined by $A_9, A_4 + A_5, A_1 + A_8 \subset T_{2,3,7}$, and $L_-(2) \hookrightarrow \Lambda_-$, we have the following.

Proposition A.9. *There exist a primitive embedding of N_G into the K3 lattice Λ such that $N_G \cap \Lambda_+(2) = L_+(2)$ and $L \cap \Lambda_- = L_-(2)$.*

Proof of Theorem 1.2. Let G be one of the three groups $G_{(6)}, G_{(9)}, G_{(10)}$ (or equivalently $\mathfrak{S}_5, N_{72}, \mathfrak{A}_6$). Since N_G is of Leech type the action G on N_G extends to that on the K3 lattice. By our construction it preserves $\Lambda_+(2)$ and Λ_- . Since $L_-(2)$ does not contain a (-2) -element, there exists an Enriques surface $S_{(n)}$ ($n = 6, 9, 10$) such that $H^{1,1}(S_{(n)}, \mathbb{Z})^- \simeq L_-(2)$ by the subjectivity theorem ([4]). Let $h \in H^2(S_{(n)}, \mathbb{Z})_f$ be a primitive element perpendicular to L_+ . h is unique up to sign. Replacing with $-h$ if necessary, we may assume that h belongs to the positive cone, that is, the connected component of $\{x \in H^2(S, \mathbb{R}) \mid (x^2) > 0\}$ which contains ample classes. There exists a composition $w \in O(H^2(S_{(n)}, \mathbb{Z})_f)$ of reflections with respect to smooth rational curves on $S_{(n)}$ such that $w(h)$ is nef. By the strong Torelli type theorem ([3]), the cohomological action of $G \curvearrowright H^2(S_{(n)}, \mathbb{Z})_f$ twisted by w is realized by an algebraic action. \square

Remark A.10. By construction and by Lemma A.1 and the Torelli type theorem, two Enriques surfaces $S_{(9)}$ and $S_{(10)}$ are isomorphic to each other. The Enriques surface $S_{(6)}$ is \mathfrak{S}_5 -equivariantly isomorphic to that of type VII in Kondo[16]. In particular, $\text{Aut } S_{(6)}$ is the symmetric group \mathfrak{S}_5 .

Remark A.11. A Mathieu action of $G = C_2 \times \mathfrak{A}_4$ on an Enriques surface can be constructed lattice theoretically also. Via the embedding $\mathfrak{S}_6 \hookrightarrow M_{12}$, G is embedded into M_{12} and decomposes Ω_{\pm} into three orbits of length 2, 4 and 6. Hence the lattice $N_{\pm, G}$ contains the root lattice of type $A_1 + A_3 + A_5$ as a sublattice of index two. $N_{+, G}$ has a primitive embedding into the Enriques lattice Λ_+ since Λ_+ contains the lattice $T_{2,4,6}$ as a sublattice of index 2. $N_{-, G}$ has a primitive embedding into $I_{2,10}$ since $N_{-, G} \simeq N_{+, G}$ and since $I_{2,10} \simeq \Lambda_+ + \langle 1 \rangle + \langle -1 \rangle$. Hence the same argument shows the existence of (a 1-dimensional family of) Enriques surfaces with Mathieu actions of G .

APPENDIX B. K3 SURFACE CONSTRUCTED IN [14, 15]

In [14, 15] Keum, Oguiso and Zhang constructed a K3 surface with an action by a group $\mathfrak{A}_6 : C_4$ and determined the abstract structure of the

group. Here we show that it contains a fixed point free involution ε and the action by \mathfrak{A}_6 descends to $S = X/\varepsilon$. Hence this gives another lattice theoretic construction of \mathfrak{A}_6 -action.

We start with recalling their results.

Theorem B.1. *There exists a K3 surface X with the following properties.*

- (1) *X is a smooth K3 surface with Picard number $\rho = 20$ and the transcendental lattice T_X is given by the Gram matrix $\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$.*
- (2) *X is acted on by a group $\widetilde{\mathfrak{A}}_6 = \mathfrak{A}_6 : C_4$. Here \mathfrak{A}_6 is the subgroup of symplectic automorphisms and satisfies $NS(X)^{\mathfrak{A}_6} = \mathbb{Z}H$, $(H^2) = 20$.*
- (3) *The image of the natural homomorphism $c: \widetilde{\mathfrak{A}}_6 \rightarrow \text{Aut}(\mathfrak{A}_6)$ is M_{10} .*

We put ε to be the nontrivial element in $\ker c$ (namely the element $(1, -1) \in M_{10} \times C_4$ in the notation of [15, Theorem 2.3]). The corresponding automorphism on X is denoted by the same letter.

Lemma B.2. *The automorphism ε is fixed point free.*

Proof. By the construction, ε is a non-symplectic involution on X , hence its fixed locus is a disjoint union of smooth curves. Assume it is not empty. We look at the divisor D given by the sum of fixed curves. Since ε commutes with \mathfrak{A}_6 , D belongs to the sublattice $NS(X)^{\mathfrak{A}_6} = \mathbb{Z}H$ by Theorem B.1 (2). Since H is ample, D is connected. Then $(H^2) = 20$ shows that the genus of D is (at least) 11, but there are no such fixed curves for non-symplectic involutions. Thus ε is free. \square

Therefore, the Enriques surface $S = X/\varepsilon$ has an action by \mathfrak{A}_6 (or by M_{10} , more precisely).

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